Matrix Model Formulation of LS-Covariant Noncommutative Quantum Field Theories on Minkowski Spacetime

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Abstract

In this thesis we construct a class of noncommutative quantum field theories on Minkowski spacetime via an analytical continuation of the Euclidean Grosse-Wulkenhaar and LSZ models, which are defined by a perturbative setting based on modified Feynman diagrams. Characterstic of these theories is the presence of a constant, external electromagnetic field, which renders their ultraviolet and infrared regimes indistinguishable. This feature is known as LS-duality and is believed to be responsible for their renormalizability and the vanishing of their β-functions in the Euclidean case.

We introduce an alternative to the $\mathrm{i}\,\epsilon$-prescription of these Minkowskian models, which will be shown to lead to causal propagators. This regularization allows us to map the LS-covariant theories onto matrix models via a generalization of the Landau basis, and to impose a simultaneous UV- and IR-regularization of the Feynman diagrams, while keeping the LS-duality manifestly. A new quality on Minkowski spacetime is the instability of the vacuum with respect to pair production, which is due to the lack of translation invariance caused by the electromagnetic field. We discuss its implication on the perturbative expansion and the unitarity of the scattering matrix. As a first step towards a renormalization of these theories, we derive the corresponding propagators in Minkowski spacetime in position and matrix representation and discuss their asymptotics.

Kurzbeschreibung

In dieser Arbeit konstruieren wir eine Klasse nichtkommutativer Quantenfeldtheorien auf Minkowski Raumzeit über analytische Fortsetzungen der euklidischen Grosse-Wulkenhaar und LSZ Modelle, welche über einen perturbativen Ansatz mit Hilfe von modifizierten Feynman Diagrammen definiert sind. Charakteristisch für diese Theorien ist die Anwesenheit eines konstanten, äußeren elektromagnetischen Feldes, welches ihre infrarot und ultraviolet Bereiche ununterscheidbar macht. Diese Symmetrie ist bekannt als *LS-Dualität*, und scheint verantwortlich zu sein für ihre Renormierbarkeit und das Verschwinden ihrer β-Funktion im Euklidischen Fall.

Wir führen eine Alternative zur $\mathrm{i}\,\epsilon$-Vorschrift für diese Modelle auf Minkowski Raumzeit ein, die, wie wir zeigen werden, ebenfalls zu kausalen Propagatoren führt. Diese Regularisierung erlaubt uns mit Hilfe einer Verallgemeinerung der Landau Basis die LS-kovarianten Modelle auf Matrix Modelle abzubilden, und eine gleichzeitige UV- und IR-Regularisierung der Feynman Diagramme durchzuführen, welche die LS-Dualität manifest erhält. Eine neue Qualität auf Minkowski-Raumzeit ist die Instabilität des Vakuums bezüglich Paar-Produktion, welche aus einem von dem elektromagnetischen Feld verursachten Fehlen der Translationsinvarianz folgt. Wir diskutieren deren Auswirkungen auf die Störungsentwicklung und die Unitarität der Streumatrix. Als einen ersten Schritt in Richtung Renormierung dieser Theorien leiten wir die zugehörigen Propagatoren in Minkowski-Raumzeit in Orts- und Matrix-Darstellung her und diskutieren ihr asymptotisches Verhalten.

Contents

1 Introduction

1.1 Motivation

Quantum field theory is a powerful framework for the description of physical phenomena, providing an astonishing agreement of theory and experiment. But despite its success, the reconciliation of quantum theory and gravity remains an open issue. A long-held belief is that an underlying theory of quantum gravity should manifest itself in a modification of the fundamental geometry at very short distances and may be accompanied by a quantization of spacetime itself.

The idea to consider theories on quantum spacetime goes back to the early days of quantum field theory. The need for a regularization at high energies led people to doubt the ordinary concept of spacetime at small scales. Inspired by quantum mechanics, where single points in phase space loose their meaning, uncertainty relations for spacetime coordinates induced by the commutation relations

$$[x^\mu, x^\nu] = \mathrm{i}\,\Theta^{\mu\nu}(\boldsymbol{x})\,, \tag{1.1}$$

should prevent the resolution of arbitrary small scales and effectively regularize the high energy divergences. However, the papers by Snyder [Sny47a, Sny47b], who published the first systematic analysis on this subject, were largely ignored, due to the enormous success of the renormalization program.

The mathematical foundation of noncommutative spacetimes has been developed by Alain Connes in form of his noncommutative geometry. As a surprise, the standard model fits quite naturally into the frame of noncommutative geometry. Using the notion of a *spectral action principle*, Connes et al. were able to deduce the standard model of particles including the Higgs mechanism (with a prediction for the Higgs mass around 170 ± 10 GeV [Sch07]) and gravitation from first principles (see e.g. [Con94, GB02, CC10]). Though it still suffers from several shortcomings, as it is (up to now) only a classical but not a quantum theory, these investigations finally directed peoples attention to noncommutative quantum field theory. A first application was found in condensed matter systems, as it seems to be the right framework to describe the fractional quantum Hall effect (see e.g. [HVR01]). After it was realized that NCQFT arises in string and M-theory [CDS98, DH98, CH99, Sch99, SW99] it gained huge popularity. It was shown that certain low-energy limits lead to an effective noncommutative Yang-Mills theory

$$\mathcal{S}_{\mathsf{YM}} = \int \mathrm{d}^4x \left(\frac{1}{4g^2} F_{\mu\nu} \star F^{\mu\nu}\right) \tag{1.2}$$

with

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \mathrm{i}\,(A_\mu \star A_\nu - A_\nu \star A_\mu)\,. \tag{1.3}$$

The product denoted by $\star$ is the *Groenewold-Moyal product*, realizing the commutation relation (1.1) with constant *deformation matrix* $\Theta^{\mu\nu}$ and noncommuting *space* coordinates. Recently it has been shown that noncommutative quantum field theory also appears as a low-energy limit in another popular approach to quantum gravity, namely loop quantum gravity [FL06, JMN09]. NCQFT might thus well be seen as a first step towards a full theory of quantum gravity.

Inspired by the noncommutative YM action, several noncommutative versions of quantum field theories have been proposed by taking the usual classical action defined on some commutative spacetime and replacing the ordinary product by the star-product with constant deformation matrix. The quantum theory is defined perturbatively via *modified Feynman rules*, which in momentum space amounts to using the ordinary Feynman propagator but with modified interaction vertices, which carry momentum depending phase factors [Fil96]. The original hope of Snyder and contemporaries, that the fuzziness of spacetime would regulate all UV divergences, soon turned out to be too optimistic. Filk showed that Feynman diagrams for the noncommutative ϕ^4-theory can be classified into planar and non-planar diagrams [Fil96]. The planar diagrams turn out to be identical to their commutative counterpart and have to be renormalized accordingly. The non-planar diagrams, on the other side, suffer from what is called *UV/IR mixing* [MVRS00], which ultimately leads to infinitely many non-renormalizable diagrams.

Soon the lack of unitarity of the corresponding S-matrix was discovered [GM00], which manifests itself in a violation of the cutting rules. It was traced back to the noncommutativity of space and time $\Theta^{0i} \neq 0$ and has found to be absent for pure space/space noncommutativity $\Theta^{0i} = 0$. This seemed to be in concordance with the fact that theories with noncommuting time and space coordinates should arise from open strings moving in an external *electric* background which, however, have no low energy effective field theory limit. As has been shown in [BDFP02], the violation of unitarity is not present in a perturbative setting using the Dyson series, involving time ordered products of the interaction Hamiltonian in the context of canonical quantization, or the Yang-Feldman formalism. The transition from the Dyson series to Feynman diagrams is usually performed with help of Wick's theorem, which necessitates reversing the order of time ordering and field multiplication. These two operation, however, do not commute if $\Theta^{0i} \neq 0$, which shows that in this case path integral quantization and canonical quantization are simply not equivalent.

Despite its apparent drawbacks, the "traditional" NCQFT on Euclidean space based on the path integral quantization has received an increased attention since the advent of the Grosse-Wulkenhaar (GW) model. The GW model was the first noncommutative model which proved to be renormalizable to all orders in perturbation theory in two [GW03] and four dimensions [GW05b]. Grosse and Wulkenhaar realized that the UV/IR mixing problem, which is the reason for the non-renormalizability of the usual noncommutative $\phi^{\star 4}$ model, is due to a missing term in the action. By adding an harmonic oscillator term and treating it non-perturbatively, the asymptotic behavior of the propagator improved such as to overcome the UV/IR mixing problem and even rendered the GW model renormalizable.

A particular surprising feature of this model is the vanishing of the β-function [GW04, DR07, DGMR07]. In four dimensions, both, the bare and the renormalized coupling constant remain bounded and non-zero after removing the UV cutoff. Thus the model has no *Landau ghost* (or *triviality problem*) and is *not* asymptotically free but *asymptotically safe*. This is contrary to the commutative case, where the only models without Landau ghost are non-Abelian gauge theories. Roughly, the problem is that even after successful renormalization some coupling parameters still may diverge at small but finite scales. Simple renormalizable theories in commutative QFT, like QED or ϕ^4 theory in 4 dimensions, are affected by this problem. It became clear that QED had to be incorporated into a larger theory where this problem no longer persist. Up to now the only commutative theories which do not suffer from the Landau problem are non-abelian gauge theories [GW73, Pol73]. The GW model is the first rigorous four dimensional field theory without unnatural cutoff, which is expected to exist non-perturbatively [Riv07a] and is not asymptotically free.

The GW breakthrough paved the way for a construction of various renormalizable NCQFT defined on Euclidean space. The crucial ingredient turned out to be the invariance under Fourier transformation plus a rescaling of the fields, known as LS-duality [LS02a]. It was incorporated into the GW model through the enhancement of the action by the extra harmonic oscillator term. The procedure of making a theory LS-covariant is now known as *vulcanization*[1] and has successfully been applied to other models, rendering them renormalizable. Among these are the $\phi^{\star 3}$-model [GS06b, GS06a, GS08], the Gross-Neveu model [VT07a] and the LSZ model [LSZ03, LSZ04].

The vulcanization of the Euclidean models had the convenient side-effect that the corresponding free parts of the action get diagonalized by a countable infinite set of functions, known as *Landau functions*. With help of this basis the LS-duality covariant models are mapped onto matrix models. The matrix approach permits an easy way of regularizing the model while keeping the *LS-duality* manifestly at quantum level. In this way, Grosse and Wulkenhaar were able to show the renormalizability of their model to all order in perturbation theory. In addition, it has been used to solve the LSZ model exactly and prove the vanishing of the β-function.

In this thesis we wish to answer the question: do the LS-duality covariant models have a counterpart on noncommutative Minkowski spacetime, and if yes, are they renormalizable? Up to now there exist only partial results in this direction. In [WW07] a complex model in three dimensions, i.e. with *degenerated deformation matrix* and thus with one commuting coordinate, based on a complex version of the Grosse

[1] Vulcanization alludes to a technological operation with the same name, which adds sulphur to rubber to improve its mechanical properties and its resistance to temperature change [Riv07b].

Wulkenhaar model with a $(\phi^\dagger\phi)^{\star 3}$-potential has been considered and proven to be renormalizable. A real $\phi^{\star 4}$ model in 4 dimensions with two commuting coordinates has been proven to be renormalizable in [GVT08]. A renormalizable NCQFT on Minkowski spacetime might thus be constructed by using renormalizable Euclidean theories equipped with a commutative time dimension, in which case the modified Feynman rules apply. We will go one step further and consider the full noncommutative Minkowski spacetime. Irrespective of the fact that the path integral quantization has been spotted to be responsible for the violation of unitarity, we will work in the usual perturbation theory. The purpose is to sound the possibility to construct a renormalizable and non-trivial four-dimensional quantum field theory in Minkowski spacetime with the help of the noncommutative deformation.

We define bosonic LS-duality covariant models in Minkowski spacetime, the LSZ and GW model, based on the work [FS09, FS10]. While for all frequently investigated Euclidean models the vulcanization procedure produces discrete "harmonic oscillator like" spectra for the wave operators which are involved, the Minkowski signature turns them to be continuous and unbounded from below. The discrete spectrum is the main ingredient for a reasonable application of the matrix basis. In the course of this thesis we will demonstrate how to overcome this barrier by a proper regularization of the model, which will be called ϑ-regularization and is a replacement for Feynman's $\mathrm{i}\,\epsilon$-prescription. As will turn out, this regularization is also connected to causality and leads to the Feynman propagator. The Feynman graphs are analytically continuations of the Euclidean ones. Comparing to recent results on the Minkowskian Grosse-Wulkenhaar model [Zah10], based on the usual $\mathrm{i}\,\epsilon$-regularization, we find that the strange divergences found in [Zah10] are absent in the matrix approach. The ϑ-regularization thus seems to be necessary to define LS-duality covariant models in Minkowski spacetime. We will also discuss the problem of unitarity of these models, which require a more careful analysis due to the lack of translation invariance and the occurrence of pair creation. The propagators of these models will be calculated and their asymptotics discussed. The ϑ-regularization turns out to improve their asymptotic behaviour and may thus turn out to be crucial for the renormalization program.

The thesis is structured as follows: In chapter 2 we give a brief introduction to path integral quantization of noncommutative field theories in Euclidean and Minkowski spacetime. We derive its modified Feynman rules and illustrate the appearance UV/IR mixing problem. Chapter 3 is devoted to the origin of the UV/IR mixing and the question how to tame it. We introduce Euclidean versions of the LS-covariant models and the translation-invariant model as examples of NCQFT without UV/IR mixing problem. In chapter 4 we give a brief account on the matrix basis, which has been an invaluable tool in the investigation of LS-covariant models on Euclidean space. A proof for LS-covariance at quantum level will be given. In chapter 5 we introduce the Minkowskian versions of bosonic LS-covariant models, the LSZ and GW model. We investigate its spectral structure and sound the possibility of a matrix representation. We point out the differences to the Euclidean models and find a representation in terms of a continuous set of eigenfunctions and a matrix representation in terms of resonances. Both approaches are related to different ways to establish the corresponding quantum field theory. In chapter 6 we give an account on the new matrix basis and derive the matrix model representation of the LS-covariant models on Minkowski spacetime. Chapter 7 is devoted to the application of the methods introduced before. We show that the matrix approach leads to causal propagators and is a natural representation to implement LS-covariance at quantum level. The unitarity problem for LS-covariant theories is touched afterwards. Finally we investigate their renormalization properties in chapter 8 by calculating the corresponding propagators and scrutinizing their asymptotic behavior.

1.2 Notation

We will shortly comment on the notation and conventions we will use in the forthcoming chapters. We will work in D-dimensional Euclidean or Minkowskian space with $D = 2n$ and $n \in \mathbb{N}$, with signatures $(1, \ldots, 1)$ and $(1, -1, \ldots, -1)$, respectively. Euclidean vectors are denoted as

$$\boldsymbol{a} = (a^i) = (a^1, \ldots, a^D) \tag{1.4}$$

and are indicated by Latin indices $i, j, \ldots$ running from 1 to D. Minkowskian vectors are denoted by

$$\boldsymbol{a} = (a^\mu) = (a^0, \ldots, a^d)\,, \tag{1.5}$$

indicated by Greek indices $\mu, \nu, \ldots$ which take values in $\{0, 1, \ldots, d = D - 1\}$. The $D = 2n$-dimensional coordinate vector $\boldsymbol{x}$ will occasionally be split up into two-dimensional subvectors

$$\boldsymbol{x} = (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)\,, \tag{1.6}$$

with $\boldsymbol{x}_k = (x^{2k-1}, x^{2k})$ in Euclidean space and $\boldsymbol{x}_k = (x^{2k-2}, x^{2k-1})$ in Minkowskian spacetime. In two dimensional Euclidean space the coordinates are often denoted as $\boldsymbol{x} = (x, y)$, whereas in two-dimensional Minkowski spacetime we write $\boldsymbol{x} = (t, x)$.

The usual Einstein convention is used to describe the scalar products with $a_i b^i$ and $a_\mu b^\mu$ denoting the products in the respective cases. If the specific signature is irrelevant or follows from the context we will simply write $\boldsymbol{a} \cdot \boldsymbol{b}$. In order to avoid notational clutter, we will introduce a special notation for the square of a vector $\boldsymbol{a}$ with respect to the different signatures. Performed with respect to Euclidean signature it reads

$$a_i^2 := a_i a^i = a_1^2 + \dots a_D^2 \,. \tag{1.7}$$

This allows us to distinguish it easily from its Minkowskian counterpart denoted as

$$a_\mu^2 := a_\mu a^\mu = a_0^2 - a_1^2 - \dots - a_d^2 \,. \tag{1.8}$$

Integrations will partly be abbreviated as

$$\int_x := \int_{\mathbb{R}^D} \mathrm{d}\boldsymbol{x} \quad \text{and} \quad \int_k := \int_{\mathbb{R}^D} \mathrm{d}\boldsymbol{k} \,. \tag{1.9}$$

We will often switch between functions $f(\boldsymbol{x})$ defined on some space and abstract "kets" $|f\rangle$, where according to Dirac's bra-ket notation we define

$$\langle \boldsymbol{x} | f \rangle = f(\boldsymbol{x}) \,, \tag{1.10}$$

where the specific representation will be clear in the given context. The L^2-scalar product of two functions $f, g \in L^2(\mathbb{R}^D)$ is then defined by

$$\langle f | g \rangle = \int_{\mathbb{R}^D} \mathrm{d}^D x \, f(\boldsymbol{x})^* \, g(\boldsymbol{x}) \,, \tag{1.11}$$

where $f(\boldsymbol{x})^*$ is the complex conjugated function of $f(\boldsymbol{x})$, sometimes also denote as $\overline{f(\boldsymbol{x})}$. As is common practice in the physical literature, this definition will freely be extended to objects like tempered distributions etc, whenever it is clear what is meant by the pairing (1.11).

The hermitian conjugation of a matrix M is designated by a dagger with $M^\dagger = (M_{mn})^\dagger = (M_{nm})^*$.

We will also use the notation

$$\begin{aligned} \boldsymbol{x} \cdot \boldsymbol{E} \cdot \boldsymbol{x}' &= x^\mu E_{\mu\nu} x'^\nu \\ \boldsymbol{x} \cdot \boldsymbol{B} \cdot \boldsymbol{x}' &= x^i B_{ij} x'^j \,, \end{aligned} \tag{1.12}$$

for $\boldsymbol{x}, \boldsymbol{x}' \in \mathbb{R}^2$ and $E_{\mu\nu}$ and B_{ij} are the two-dimensional electric and magnetic field strengths, respectively, defined as

$$\boldsymbol{E} = (E_{\mu\nu}) = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix} \quad , \quad \boldsymbol{B} = (B_{ij}) = \begin{pmatrix} 0 & B \\ -B & 0 \end{pmatrix} \tag{1.13}$$

with $E, B > 0$.

The Fourier transformation of a function f is defined as

$$\hat{f}(\boldsymbol{k}) = \frac{1}{(2\pi)^{D/2}} \int_{\mathbb{R}^D} \mathrm{d}^D \boldsymbol{x} \, \mathrm{e}^{-\mathrm{i}\, \boldsymbol{k} \cdot \boldsymbol{x}} f(\boldsymbol{x}) \,, \tag{1.14}$$

where the signature within the scalar product will be clear from the context. It will sometimes also be denoted as $\mathcal{F}[f]$.

Furthermore we define

$$\begin{aligned} \mathbb{R}_+ &= \{x \in \mathbb{R} \mid x \geq 0\} \\ \mathbb{R}_+^n &= \{\boldsymbol{x} = (x^1, \dots, x^n) \in \mathbb{R}^n \mid x^i \geq 0 \; \forall i\} \\ \mathbb{C}_+ &= \{z \in \mathbb{C} \mid \mathfrak{Re}(z) \geq 0\} \,. \end{aligned} \tag{1.15}$$

We define the map $(\cdot, \cdot)_\vartheta : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{C}$ for $\vartheta \in [-\pi/2, \pi/2]$ by

$$(\boldsymbol{x}, \boldsymbol{x}')_\vartheta = \cos(\vartheta) \, (\boldsymbol{x}, \boldsymbol{x}')_E + \mathrm{i} \, \sin(\vartheta) \, (\boldsymbol{x}, \boldsymbol{x}')_M \,, \tag{1.16}$$

where $(\cdot,\cdot)_M$ is the two dimensional Minkowskian and $(\cdot,\cdot)_E$ the two dimensional Euclidean scalar product. In addition we define the map $\|\cdot\| : \mathbb{R}^2 \to \mathbb{C}$ by

$$\begin{aligned} \|\boldsymbol{x}\|_\vartheta^2 &= (\boldsymbol{x},\boldsymbol{x})_\vartheta \\ &= \cos(\vartheta)\|\boldsymbol{x}\|_E^2 + \mathrm{i}\,\sin(\vartheta)\|\boldsymbol{x}\|_M^2 \end{aligned} \tag{1.17}$$

with $\|\cdot\|_E$ the two dimensional Euclidean and $\|\cdot\|_M$ the two dimensional Minkowskian norm.

2 Noncommutative Quantum Field Theories

This chapter is intended as a brief introduction to "ordinary", i.e. non-LS-covariant, noncommutative quantum field theories in the path integral framework and their shortcomings. The definition of NCQFTs consists of two independent steps, the introduction of a noncommutative spacetime and the quantization of physical fields. These two steps do not commute, so there are initially two different ways to proceed. The standard procedure amounts to first define functions on a deformed spacetime which in our case will be the Moyal space. The way we do this is known as Weyl "quantization" illustrated in the next section. Path integral quantization of the classical noncommutative field theory will be defined in section 2.2. We will discuss the problems of ordinary theories in the path integral framework using the example of the bosonic $\phi^{\star 4}$-theory. We derive the related modified Feynman rules in Euclidean and Minkowskian case, explain the UV/IR mixing problem and the unitarity problem in Minkowski spacetime.

2.1 Moyal Space and Weyl Quantization

The following discussion is valid for both Minkowskian and Euclidean signature. For convenience we will stick to D-dimensional Minkowski spacetime with D even. The Euclidean version may be obtained by using Euclidean instead of Minkowskian scalar products.

We are searching for a realization of a classical field theory defined a noncommutative space where the noncommuting coordinates obey the commutation relations

$$[x^\mu, x^\nu] = \mathrm{i}\,\Theta^{\mu\nu}\,. \tag{2.1}$$

In the following $\Theta^{\mu\nu}$ will be fixed to a constant, real-valued and antisymmetric and non-degenerate $D \times D$ matrix, known as *Moyal deformation*.[1] Its entries have the dimension of $(\text{length})^2$. A constant deformation matrix distinguishes several directions in spacetime and thus implies the breaking of Lorentz invariance (or $SO(D)$-invariance in case of Euclidean metric).[2] Similar to the electromagnetic field tensor in Maxwell theory the deformation parameter in the Minkowski case has a "magnetic part" given by Θ^{ij} for $i,j = 1,\ldots,d$ measuring space/space noncommutativity and an "electric part" Θ^{0i} for $i = 1,\ldots,d$ responsible for time/space noncommutativity. New phenomena like the loss of unitarity and the inequivalence of different quantization methods can be traced back to the latter.

A natural way of implementing a noncommutative space is to replace spacetime coordinates x^μ in $\mathbb{R}^D$ by Hermitian operators $\hat{\boldsymbol{x}}^\mu$ defined on some Hilbert space $\mathcal{H}$. The $\hat{\boldsymbol{x}}^\mu$ generate a Banach *-algebra which is isomorphic to $\mathbb{R}^D_\Theta$, which is the ring of formal power series $\mathbb{C}[[x_1,\ldots,x_D]]$ modulo the ideal generated by $x^\mu x^\nu - x^\nu x^\mu - \Theta^{\mu\nu}$. In order to define field theories on $\mathbb{R}^D_\Theta$ we need functions on this space. The *Schwartz space* $\mathcal{S}(\mathbb{R}^D)$ is defined as the set of all smooth and complex-valued functions $f : \mathbb{R}^D \to \mathbb{C}$ obeying

$$\sup_{\boldsymbol{x}} (1+|\boldsymbol{x}|)^{k+n_0+\cdots+n_d} \left|\partial_0^{n_0}\cdots\partial_d^{n_d} f(\boldsymbol{x})\right|^2 < \infty \tag{2.2}$$

for every set of integers $k, n_i \in \mathbb{N}$. The transition from ordinary Schwartz functions to functions on $\mathbb{R}^D_\Theta$ demands an ordering prescription for products of operators. The so called *Weyl ordering* is imposed by Fourier expanding the function and replacing the occurring plane waves by its operator counterpart $U(\boldsymbol{k}) =$

[1] In general Θ might be any function depending on the coordinates with $\Theta_{\mu\nu} = -\Theta_{\nu\mu}$, satisfying the Jacobi identity. The "Lie-algebra case" $\Theta^{\mu\nu} = \lambda^{\mu\nu}_\sigma x^\sigma$ with complex structure constant $\lambda^{\mu\nu}_\sigma$ leads to fuzzy and κ-deformed spaces. A third popular choice is the "quadratic case" with $\Theta^{\mu\nu} = -\mathrm{i}\left(\frac{1}{q}R^{\mu\nu}_{\rho\sigma} - \delta^\mu_\sigma\delta^\nu_\rho\right)x^\rho x^\sigma$ which leads to the definition of quantum groups.

[2] There are approaches to noncommutative field theories which avoid the breaking of Lorentz invariance at this level by choosing $\Theta^{\mu\nu}$ to be a central operator encompassed by a whole spectrum of matrices connected by Lorentz transformations. In these models, known as DFR models [DFR95, Bah04, Pia10], however, Lorentz invariance gets broken by the definition of the interactions.

$\mathrm{e}^{\,\mathrm{i}\,k_\mu \hat{\boldsymbol{x}}^\mu}$. This procedure is called *Weyl quantization* [Wey50]. At the heart of this quantization lies the relation

$$\mathrm{e}^{\,\mathrm{i}\,k_\mu \hat{\boldsymbol{x}}^\mu}\,\mathrm{e}^{\,\mathrm{i}\,p_\mu \hat{\boldsymbol{x}}^\mu} = \mathrm{e}^{\,\mathrm{i}\,(k_\mu+p_\mu)\,\hat{\boldsymbol{x}}^\mu}\,\mathrm{e}^{-\frac{\mathrm{i}}{2}k_\mu\Theta^{\mu\nu}p_\nu}\,, \tag{2.3}$$

which can easily be obtained from the Campbell-Baker-Hausdorff-formula and equation (2.1). The *Weyl-Heisenberg group* is generated by the elements $U(\boldsymbol{k}) = \mathrm{e}^{\,\mathrm{i}\,k_\mu \hat{\boldsymbol{x}}^\mu}$ and the exponential $\mathrm{e}^{-\mathrm{i}\,k_\mu\Theta^{\mu\nu}p_\nu}$ is referred to as *twisting*.

Given a Schwartz function f its *Weyl symbol* is thus given by

$$\hat{\mathcal{W}}\,[f] = \frac{1}{(2\pi)^D}\int_{\mathbb{R}^D} \mathrm{d}^D\boldsymbol{k}\,\hat{f}(\boldsymbol{k})\;\mathrm{e}^{\,\mathrm{i}\,k_\mu\,\hat{\boldsymbol{x}}^\mu}\,. \tag{2.4}$$

where $\hat{f}$ denotes the Fourier transformed field defined as in (1.14). The mapping (2.4) depends on the deformation matrix Θ through the relation (2.3). One can write (2.4) as

$$\hat{\mathcal{W}}\,[f] = \int \mathrm{d}^D\boldsymbol{x}\,f(\boldsymbol{x})\,\hat{\Delta}(\boldsymbol{x})\,, \tag{2.5}$$

where we introduced the Hermitian operator $\hat{\Delta}(\boldsymbol{x})$

$$\hat{\Delta}(\boldsymbol{x}) = \int \frac{\mathrm{d}^D\boldsymbol{k}}{(2\pi)^D}\,\mathrm{e}^{\,\mathrm{i}\,k_\mu\cdot(\hat{\boldsymbol{x}}^\mu - x^\mu)}\,. \tag{2.6}$$

The $\hat{\Delta}(\boldsymbol{x})$ serve as a mixed basis for operators and fields on spacetime. In the commutative case, i.e. $\Theta^{\mu\nu} = 0$, the exponential factorizes leading to the simple relation $\hat{\Delta}(\boldsymbol{x}) = \delta^D(\hat{\boldsymbol{x}}^\mu - x^\mu)$. The usual integral is replaced by the trace on the Hilbert space $\mathcal{H}$. Normalized as

$$\mathrm{Tr}\,\hat{\mathcal{W}}\,[f] = \int \mathrm{d}^D\boldsymbol{x}\,f(\boldsymbol{x})\,, \tag{2.7}$$

the $\hat{\Delta}(\boldsymbol{x})$ form an orthonormal set with respect to this trace

$$\mathrm{Tr}[\hat{\Delta}(\boldsymbol{x})\hat{\Delta}(\boldsymbol{y})] = \delta^D(\boldsymbol{x}-\boldsymbol{y})\,. \tag{2.8}$$

The Weyl-Heisenberg algebra has a faithful representation on the space of Weyl symbols. However, we will also need a representation in terms of the original Schwartz functions. Due to (2.8) the transformation $f \mapsto \hat{\mathcal{W}}\,[f]$ is invertible with inverse given by

$$f(\boldsymbol{x}) = \mathrm{Tr}\,[\,\hat{\mathcal{W}}\,[f]\,\hat{\Delta}(\boldsymbol{x})\,] =: \mathsf{W}[\hat{\mathcal{W}}\,[f]](\boldsymbol{x})\;, \tag{2.9}$$

dubbed as *Wigner distribution function* of the operator $\hat{\mathcal{W}}\,[f]$ [Wig32]. We will especially need the the explicit form of Wigner transformation in 1+1 dimensions corresponding to the deformation parameter $\Theta^{01} = \theta$, which for an operator $\hat{\boldsymbol{\rho}}$ is given by

$$\mathsf{W}\,[\hat{\boldsymbol{\rho}}] = \int \mathrm{d}k\,\mathrm{e}^{\,\mathrm{i}\,kx^1/\theta}\langle x^0 + k/2|\hat{\boldsymbol{\rho}}|x^0 - k/2\rangle\,. \tag{2.10}$$

One can show that [Sza03]

$$\hat{\Delta}(\boldsymbol{x})\hat{\Delta}(\boldsymbol{y}) = \frac{1}{\pi^D\det\Theta}\int \mathrm{d}^D\boldsymbol{z}\,\hat{\Delta}(\boldsymbol{z})\,\mathrm{e}^{-2\,\mathrm{i}\,(\boldsymbol{x}-\boldsymbol{z})\cdot\Theta^{-1}\cdot(\boldsymbol{y}-\boldsymbol{z})}\,. \tag{2.11}$$

from which we immediately conclude

$$\hat{\mathcal{W}}\,[f]\,\hat{\mathcal{W}}\,[g] = \int \mathrm{d}^D\boldsymbol{z}\,(f\star_\Theta g)(\boldsymbol{z})\hat{\Delta}(\boldsymbol{z}) = \hat{\mathcal{W}}\,[f\star_\Theta g]\;. \tag{2.12}$$

with

$$(f\star_\Theta g)(\boldsymbol{x}) := \frac{1}{\pi^D|\det\Theta|}\int_{\mathbb{R}^D}\mathrm{d}^D\boldsymbol{y}\,\mathrm{d}^D\boldsymbol{z}\,f(\boldsymbol{x}+\boldsymbol{y})\,g(\boldsymbol{x}+\boldsymbol{z})\;\mathrm{e}^{-2\,\mathrm{i}\,\boldsymbol{y}\cdot\Theta^{-1}\boldsymbol{z}}\,. \tag{2.13}$$

The product $\star_\Theta$ of arbitrary Schwartz functions $f(\boldsymbol{x})$, $g(\boldsymbol{x})$ is known as *Groenewold-Moyal product* [Gro46, Moy49]. We will simply call it star-product and often suppress the dependence on Θ by using $\star$ instead of $\star_\Theta$. We thus have a one-to-one correspondence between the space of Wigner distributions and its Weyl symbols such that the operator product of Weyl symbols is equivalent to the star product of their corresponding Wigner distributions:

$$\hat{\mathcal{W}}[f]\,\hat{\mathcal{W}}[g] \;=\; \hat{\mathcal{W}}[f \star_\Theta g] \qquad \text{and} \qquad \mathsf{W}[\hat{\mathbf{f}}\,] \star_\Theta \mathsf{W}[\hat{\mathbf{g}}] \;=\; \mathsf{W}[\hat{\mathbf{f}}\,\hat{\mathbf{g}}] \tag{2.14}$$

for arbitrary Weyl symbols $\hat{\mathbf{f}}$, $\hat{\mathbf{g}}$. One can show that it is associative, but not commutative

$$\begin{gathered}(f \star_\Theta (g \star_\Theta h)) = ((f \star_\Theta g) \star_\Theta h) \\ f \star_\Theta g \neq g \star_\Theta f \quad .\end{gathered} \tag{2.15}$$

As can be seen by (2.13), the product depends on the functions in a non-local manner, which has far-reaching physical consequences. Very important is the *trace property* of the integral given by

$$\int \mathrm{d}^D\boldsymbol{x}\,(f \star_\Theta g)(\boldsymbol{x}) = \int \mathrm{d}^D\boldsymbol{x}\, f(\boldsymbol{x})\, g(\boldsymbol{x}) = \int \mathrm{d}^D\boldsymbol{x}\,(g \star_\Theta f)(\boldsymbol{x})\,. \tag{2.16}$$

For analytic functions, the star product can be written in a perturbative way, called *Moyal expansion*

$$(f \star_\Theta g)(\boldsymbol{x}) = \exp\left(\frac{\mathrm{i}}{2}\Theta^{\mu\nu}\partial_\mu\partial'_\nu\right) f(\boldsymbol{x})g(\boldsymbol{x}')\Big|_{\boldsymbol{x}=\boldsymbol{x}'}\,, \tag{2.17}$$

with $\partial_\mu = \partial/\partial x^\mu$ and $\partial'_\mu = \partial/\partial x'^\mu$. It should be noted that for arbitrary functions the product (2.17) is generally not equivalent to (2.13). For a thorough investigation on the equivalence of both definitions see [EGBV89].

The space $\mathcal{S}(\mathbb{R}^D)$ equipped with the star-product is denoted by $\mathcal{A}_\Theta$. With the involution $f \mapsto f^*$ this is an associative *-algebra. By duality we can extend the star product to the space of *tempered distributions* $\mathcal{S}'(\mathbb{R}^D)$, which is the dual space of $\mathcal{S}(\mathbb{R}^D)$, consisting of all continuous functionals on $\mathcal{S}(\mathbb{R}^D)$. For $T \in \mathcal{S}'(\mathbb{R}^D)$ and $f \in \mathcal{S}(\mathbb{R}^D)$ we set

$$\langle T, f\rangle = T(f)\,. \tag{2.18}$$

Then for any $g \in \mathcal{S}(\mathbb{R}^D)$ we define the products $T \star f$ and $f \star T$ through

$$\begin{aligned}\langle T \star f, g\rangle &= \langle T, f \star g\rangle \\ \langle f \star T, g\rangle &= \langle T, g \star f\rangle\end{aligned} \tag{2.19}$$

In this way we can deal with distributions, which naturally appear in quantum field theory.

Applications to quantum field theory necessitates a relaxation of the restriction to Schwartz functions. The *multiplier algebra* $\mathcal{M} = \mathcal{M}_L \cap \mathcal{M}_R$ with $\mathcal{M}_L$ and $\mathcal{M}_R$ defined by

$$\begin{aligned}\mathcal{M}_L &= \{T \in \mathcal{S}'(\mathbb{R}^D) \,:\, \forall\, f \in \mathcal{S}(\mathbb{R}^D)\,, T \star f \in \mathcal{S}(\mathbb{R}^D)\} \\ \mathcal{M}_R &= \{T \in \mathcal{S}'(\mathbb{R}^D) \,:\, \forall\, f \in \mathcal{S}(\mathbb{R}^D)\,, f \star T \in \mathcal{S}(\mathbb{R}^D)\}\,.\end{aligned} \tag{2.20}$$

is a natural enhancement of $\mathcal{A}_\Theta$. One can show that $\mathcal{M}$ is an associative *-algebra, containing the identity, polynomials, the delta-function and its derivatives such as plane waves [GBV88]. Since the coordinates x^μ are not elements of $\mathcal{A}_\Theta$ the commutator relation

$$x^\mu \star_\Theta x^\nu - x^\nu \star_\Theta x^\mu = \mathrm{i}\,\Theta^{\mu\nu}\,. \tag{2.21}$$

does not hold in $\mathcal{A}_\Theta$ but in $\mathcal{M}$. It should be noted that an axiomatic construction of noncommutative quantum field theories analogously to the case of ordinary quantum field theory in terms of Wightman axioms is not available yet. There are hints that the framework of tempered distributions is too restrictive for the non-perturbative study of NCQFT [AGVM03]. In [Sol07b, CMTV08] the *Gel'fand-Shilov spaces* $\mathcal{S}^\beta_\alpha(\mathbb{R}^D)$ have been proposed for a enlarged framework (see appendix C.1 for a brief introduction). The corresponding multiplier algebra has been investigated in [Sol10]. In the following we will not be concerned about the right domain for a mathematical rigorous definition of NCQFTs. Nevertheless we will discuss these spaces in the context of expansion theorems for the generalized matrix basis which will be constructed in chapter 6.

In order to define physical quantities like an action we need to define integral and differentiation operations on $\mathcal{M}$ and the space of Weyl operators. The usual integral can be defined on $\mathcal{M}$ which has the trace (2.7) on $\mathcal{H}$ as its counterpart on the Weyl side. Concerning the derivatives we have at least two different natural possibilities. The ordinary derivatives defined on usual differentiable functions also define derivatives on $\mathcal{M}$

$$\partial_\mu(f \star_\Theta g) = (\partial_\mu f) \star_\Theta g + f \star_\Theta (\partial_\mu g) \,. \tag{2.22}$$

Note that they have the representation[3]

$$\partial_\mu f = [-\mathrm{i}\,(\Theta^{-1})_{\mu\nu} x^\nu, f]_\star \,. \tag{2.23}$$

This gives us a derivative on the Weyl side through $\hat{\partial}_\mu := \hat{\mathcal{W}}\left[-\mathrm{i}\,(\Theta^{-1})_{\mu\nu} x^\nu\right]$ which is an anti-Hermitian linear derivation with

$$[\hat{\partial}_\mu, \hat{\boldsymbol{x}}^\nu] = \delta_\mu^{\ \nu} \quad , \quad [\hat{\partial}_\mu, \hat{\partial}_\nu] = 0 \,. \tag{2.24}$$

One can then show that

$$[\hat{\partial}_\mu, \hat{\Delta}(\boldsymbol{x})] = -\partial_\mu \hat{\Delta}(\boldsymbol{x}) \tag{2.25}$$

and hence by partial integration

$$[\hat{\partial}_\mu, \hat{\mathcal{W}}[f]] = \int \mathrm{d}^D\boldsymbol{x}\, \partial_\mu f(\boldsymbol{x})\, \hat{\Delta}(\boldsymbol{x}) = \hat{\mathcal{W}}[\partial_\mu f] \,, \tag{2.26}$$

which proves the compatibility of both derivatives.

An interesting and in retrospective very important alternative to the usual differentiation was proposed by Filk in [Fil90]. The Weyl-Heisenberg group defined by (2.3) is a central extension of the D-dimensional group of translations:

$$U(\boldsymbol{k})U(\boldsymbol{p}) = \mathrm{e}^{-\frac{\mathrm{i}}{2} k_\mu \Theta^{\mu\nu} p_\nu} U(\boldsymbol{k}+\boldsymbol{p}) \,, \tag{2.27}$$

with $U(\boldsymbol{k}) = \mathrm{e}^{\,\mathrm{i}\, k_\mu \hat{\boldsymbol{x}}^\mu}$. Filk now proposes to consider the U's as translation operators on the deformed space and mimic the definition of a derivative in terms of the U's. The deformed translation operation on the symbol $\hat{\Delta}(\boldsymbol{x})$ defined by (2.6) is given by

$$\begin{aligned} U(\boldsymbol{k})\hat{\Delta}(\boldsymbol{x}) &= \int \frac{\mathrm{d}^D\boldsymbol{p}'}{(2\pi)^D}\, \mathrm{e}^{\,\mathrm{i}\, p'_\mu(\hat{\boldsymbol{x}}^\mu - x^\mu + \frac{1}{2}\Theta^{\mu\nu}k_\nu) + \mathrm{i}\, k_\mu x^\mu} \\ &= \mathrm{e}^{\,\mathrm{i}\, k_\mu x^\mu}\, \hat{\Delta}(\boldsymbol{x} - \frac{1}{2}\Theta \cdot \boldsymbol{k}) \,. \end{aligned} \tag{2.28}$$

and gives rise to a "covariant derivative" of $\hat{\Delta}(\boldsymbol{x})$ into the direction $\bar{\mu}$

$$\begin{aligned} \hat{\mathsf{D}}_{\bar{\mu}} \hat{\Delta}(\boldsymbol{x}) &= \lim_{\varepsilon \to 0} \frac{U(\varepsilon\, \boldsymbol{e}_{\bar{\mu}}) - U(-\varepsilon\, \boldsymbol{e}_{\bar{\mu}})}{2\varepsilon} \hat{\Delta}(\boldsymbol{x}) \\ &= \lim_{\varepsilon \to 0} \frac{\mathrm{e}^{\,\mathrm{i}\,\varepsilon\, x_{\bar{\mu}}} \hat{\Delta}(\boldsymbol{x} - \varepsilon\frac{1}{2}\Theta \cdot \boldsymbol{e}_{\bar{\mu}}) - \mathrm{e}^{-\mathrm{i}\,\varepsilon\, x_{\bar{\mu}}} \hat{\Delta}(\boldsymbol{x} + \varepsilon\frac{1}{2}\Theta \cdot \boldsymbol{e}_{\bar{\mu}})}{2\varepsilon} \\ &= \lim_{\varepsilon \to 0} \frac{\hat{\Delta}(\boldsymbol{x} - \varepsilon\frac{1}{2}\Theta \cdot \boldsymbol{e}_{\bar{\mu}}) - \hat{\Delta}(\boldsymbol{x} + \varepsilon\frac{1}{2}\Theta \cdot \boldsymbol{e}_{\bar{\mu}})}{2\varepsilon} - \mathrm{i}\, x_{\bar{\mu}} \hat{\Delta}(\boldsymbol{x}) \,, \end{aligned} \tag{2.29}$$

with $\boldsymbol{e}_{\bar{\mu}}$ being the unit vector in this direction. In terms of Wigner distributions and in Fourier space this construction yields a covariant derivative $\partial/\partial k_{\bar{\mu}} - 2\mathrm{i}\,(\Theta^{-1})_{\bar{\mu}\nu} k^\nu$. This may not be surprising, as one can think of the operators U as the parallel transport operators acting on the line bundle of fields ϕ over the plane with connection form $2(\Theta^{-1})_{\mu\nu} k^\nu$.

We thus have two different possibilities to define a classical action on a noncommutative space, using the star-product instead of the usual pointwise product but leaving the derivatives unaltered, or using the star-product and the covariant derivatives. The former approach has been the first choice, but led to severe difficulties as UV/IR mixing and nonrenormalizability, as will be explained in the next section. The second approach is a special case of a variety of renormalizable, noncommutative Euclidean quantum field theories, in the following called *LS-covariant models* and introduced in chapter 3.

[3]Infinitesimal translations are thus given by *inner derivatives*, which is in clear distinction to commutative field theories.

2.2 Quantum Field Theory

If space and time do not commute, canonical quantization, path integral quantization and Yang-Feldman quantization are no longer equivalent [BDFP02]. In the following we will give an introduction to the "traditional" Euclidean NCQFT defined through path integrals. It is certainly the most studied setup and has achieved a lot of progress in the last ten years. Afterwards we will explain its counterpart on Minkowski spacetime, outline its disadvantages and differences to other popular approaches.

2.2.1 Standard Perturbative Setting in Euclidean Space

The standard way to obtain a field theory on noncommutative Euclidean spacetime is to start with a classical action and to substitute the usual pointwise products by the star-product keeping the usual derivatives. As a simple model one may consider the $\phi^{\star 4}$ model given by

$$\mathcal{S} = \int_x \left(\frac{1}{2}\partial_i\phi \star \partial^i\phi + \frac{m^2}{2}\phi\star\phi + \frac{\lambda}{4!}\,\phi\star\phi\star\phi\star\phi\right)(\boldsymbol{x}) \tag{2.30}$$

for real fields $\phi(\boldsymbol{x})$ and $\int_x = \int_{\mathbb{R}^D} \mathrm{d}^D\boldsymbol{x}$. The trace property (2.16) implies that the free part, i.e the part of the action quadratic in the fields, is identical to the commutative one. We are thus working with an ordinary commutative field theory with "strange" interactions. Functional integral quantization will be performed by introducing a generating functional

$$Z[J] = \mathcal{N}\int \mathcal{D}\phi\, \exp\left(-\mathcal{S} + \int_x J(\boldsymbol{x})\phi(\boldsymbol{x})\right) \tag{2.31}$$

with normalization constant $\mathcal{N}$ and $\mathcal{D}\phi$ being the ordinary path integral measure of the commutative case. In the following we will show, that the generating functional defined through (2.31) can be expressed in a similar perturbative expansion as in the commutative case, leading to the usual Feynman diagrams with the star-product standing in for the ordinary pointwise product.

The free part of the generating functional

$$Z_0[J] := Z[J]_{\lambda=0} \tag{2.32}$$

fulfills the same differential equation as in the commutative case. The construction is as follows. Since the integrand vanishes at the boundaries, by partial integration we get the identity

$$\begin{aligned} 0 &= \int \mathcal{D}\phi \frac{\delta}{\delta\phi(\boldsymbol{y})} \exp\left(-\mathcal{S}_0 + \int_x J(\boldsymbol{x})\phi(\boldsymbol{x})\right) \\ &= \int \mathcal{D}\phi \left(-\frac{\delta\mathcal{S}_0}{\delta\phi(\boldsymbol{y})} + J(\boldsymbol{y})\right)\left[\exp\left(-\mathcal{S}_0 + \int_x J(\boldsymbol{x})\phi(\boldsymbol{x})\right)\right]. \end{aligned} \tag{2.33}$$

with $\mathcal{S}_0 = \mathcal{S}|_{\lambda=0}$. Noticing that for a generic functional F

$$\int \mathcal{D}\phi\, F(\phi)\, \exp\left(-\mathcal{S}_0 + \int_x J(\boldsymbol{x})\phi(\boldsymbol{x})\right) = F\left(\frac{\delta}{\delta J}\right)\int \mathcal{D}\phi \exp\left(-\mathcal{S}_0 + \int_x J(\boldsymbol{x})\phi(\boldsymbol{x})\right) \tag{2.34}$$

we arrive at the differential equation for the free generating functional

$$(-\partial_i^2 + m^2)\frac{\delta Z_0[J]}{\delta J(\boldsymbol{x})} = J(\boldsymbol{x})Z_0[J] \tag{2.35}$$

As can easily be checked this equation is solved by

$$Z_0[J] = \exp\left(\frac{1}{2}\int_x\int_y J(\boldsymbol{x})\Delta(\boldsymbol{x}-\boldsymbol{y})J(\boldsymbol{y})\right) \tag{2.36}$$

with the usual free propagator $\Delta(\boldsymbol{x}) = \langle\boldsymbol{x}|(-\partial_i^2 + m^2)^{-1}|0\rangle$.

Now we consider the full interacting theory. By partial integration we find

$$0 \quad = \quad \int \mathcal{D}\phi \left(-\frac{\delta \mathcal{S}}{\delta \phi(\boldsymbol{y})} + J(\boldsymbol{y}) \right) \left[\exp \left(-\mathcal{S} + \int_x J(\boldsymbol{x}) \phi(\boldsymbol{x}) \right) \right] . \tag{2.37}$$

We would like to pull out the terms in the first bracket in (2.37) to obtain a functional equation for $Z[J]$ analogously to (2.35). So we need an expression for the functional derivative of $\mathcal{S}$ which now contains star-products. Using the trace property and associativity of the star-product one easily shows that

$$\frac{\delta \mathcal{S}}{\delta \phi(\boldsymbol{x})} = (-\partial_i^2 + m^2)\phi(\boldsymbol{x}) + \frac{\lambda}{3!}(\phi \star \phi \star \phi)(\boldsymbol{x}) \tag{2.38}$$

while pulling this out of the functional integral leads to the differential equation [MSJ01]

$$(-\partial_i^2 + m^2)\frac{\delta Z[J]}{\delta J(\boldsymbol{x})} + \frac{\lambda}{3!} \left(\frac{\delta}{\delta J(\boldsymbol{x})} \star \frac{\delta}{\delta J(\boldsymbol{x})} \star \frac{\delta}{\delta J(\boldsymbol{x})} \right) Z[J] = J(\boldsymbol{x}) Z[J] \,. \tag{2.39}$$

The star-product of functional derivatives is a short-hand notation for

$$\mathrm{e}^{\,\frac{\mathrm{i}}{2}\Theta^{\mu\nu}\partial_{\xi_\mu}\partial_{\eta_\nu}}\, \mathrm{e}^{\,\frac{\mathrm{i}}{2}\Theta^{\rho\sigma}\partial_{\alpha_\rho}\partial_{\beta_\sigma}} \frac{\delta}{\delta J(\boldsymbol{x}+\boldsymbol{\xi})} \frac{\delta}{\delta J(\boldsymbol{x}+\boldsymbol{\eta}+\boldsymbol{\alpha})} \frac{\delta}{\delta J(\boldsymbol{x}+\boldsymbol{\eta}+\boldsymbol{\beta})} Z[J] \bigg|_{\boldsymbol{\xi}=\boldsymbol{\eta}=\boldsymbol{\alpha}=\boldsymbol{\beta}=0} . \tag{2.40}$$

We will now show, that analogously to the commutative case, the solution is given by

$$Z[J] = \mathcal{N} \exp \left(-\mathcal{S}_{\mathsf{int}} \left[\frac{\delta}{\delta J} \right] \right) Z_0[J] \tag{2.41}$$

with $\mathcal{S}_{\mathsf{int}}[\phi] = \frac{\lambda}{4!} \int_x \phi^{\star 4}$ the interaction term and $\int_x (\delta/\delta J)^{\star 4} Z[J]$ defined through (2.40) and the trace property. Using the trace property one finds

$$\left[\int_y \left(\frac{\delta}{\delta J(\boldsymbol{y})} \right)^{\star 4}, J(\boldsymbol{x}) \right] = \left(\frac{\delta}{\delta J(\boldsymbol{x})} \right)^{\star 3} . \tag{2.42}$$

Now Campbell-Baker-Hausdorff

$$\mathrm{e}^{\,A} B \,\mathrm{e}^{\,-A} = B + [A, B] + \frac{1}{2!}[A, [A, B]] + \dots \tag{2.43}$$

and the fact that $[\int (\delta/\delta J)^{\star 4}, (\delta/\delta J)^{\star 3}] = 0$ imply

$$\exp \left(\mathcal{S}_{\mathsf{int}} \left[\frac{\delta}{\delta J} \right] \right) J(\boldsymbol{x}) \exp \left(-\mathcal{S}_{\mathsf{int}} \left[\frac{\delta}{\delta J} \right] \right) = J(\boldsymbol{x}) + \mathcal{L}'_{\mathsf{int}} \left[\frac{\delta}{\delta J} \right] \tag{2.44}$$

where $\int_x \mathcal{L}_{\mathsf{int}}[\phi] = \mathcal{S}_{\mathsf{int}}[\phi]$ is the interaction Lagrangian. Putting (2.41) into (2.39) we find

$$\begin{aligned}
&(-\partial_i^2 + m^2)\frac{\delta Z[J]}{\delta J(\boldsymbol{x})} + \frac{\lambda}{3!} \left(\frac{\delta}{\delta J(\boldsymbol{x})} \star \frac{\delta}{\delta J(\boldsymbol{x})} \star \frac{\delta}{\delta J(\boldsymbol{x})} \right) Z[J] \\
\overset{(2.41)}{=} \quad & \mathcal{N} \exp \left(-\mathcal{S}_{\mathsf{int}} \left[\frac{\delta}{\delta J} \right] \right) \left((-\partial_i^2 + m^2) \frac{\delta}{\delta J(\boldsymbol{x})} + \mathcal{L}'_{\mathsf{int}} \left[\frac{\delta}{\delta J} \right] \right) Z_0[J] \\
\overset{(2.35)}{=} \quad & \mathcal{N} \exp \left(-\mathcal{S}_{\mathsf{int}} \left[\frac{\delta}{\delta J} \right] \right) \left(J(\boldsymbol{x}) + \mathcal{L}'_{\mathsf{int}} \left[\frac{\delta}{\delta J} \right] \right) Z_0[J] \\
\overset{(2.44)}{=} \quad & J(\boldsymbol{x}) \mathcal{N} \exp \left(-\mathcal{S}_{\mathsf{int}} \left[\frac{\delta}{\delta J} \right] \right) Z_0[J] \\
\overset{(2.39)}{=} \quad & J(\boldsymbol{x}) Z[J] \,,
\end{aligned} \tag{2.45}$$

which proves that (2.41) is indeed a solution for the generating functional. This can be evaluated perturbatively in terms of Feynman diagrams corresponding to $\mathcal{S}_{\mathsf{int}}$. Contrary to the usual commutative theories, the propagators are multiplied with respect to the star-product, for which this diagrammatic expansion is known as *modified Feynman rules*. These are illustrated in the next section.

2.2.2 Feynman Diagrams, UV/IR Mixing and Renormalization

Using Fourier transformation and the Campbell-Baker-Hausdorff relation, one can deduce the following momentum space representation for the $\phi^{\star 4}$ interaction part

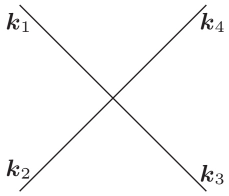
$$\int \mathrm{d}^D \boldsymbol{x}\, \phi^{\star 4}(\boldsymbol{x}) \quad = \quad \prod_{a=1}^{4} \left(\int \frac{\mathrm{d}^D \boldsymbol{k}_a}{(2\pi)^{D/2}} \right) \hat{\phi}(\boldsymbol{k}_1)\hat{\phi}(\boldsymbol{k}_2)\hat{\phi}(\boldsymbol{k}_3)\hat{\phi}(\boldsymbol{k}_4)\, \hat{V}(\boldsymbol{k}_1, \ldots, \boldsymbol{k}_4) \tag{2.46}$$

with

$$\hat{V}(\boldsymbol{k}_1, \ldots, \boldsymbol{k}_m) = (2\pi)^D \delta \left(\sum_{i=1}^{m} \boldsymbol{k}_i \right) \exp(-\,\mathrm{i} \sum_{i<j=1}^{m} \boldsymbol{k}_i \times \boldsymbol{k}_j) \tag{2.47}$$

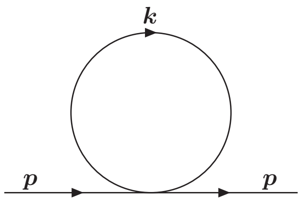
the interaction vertex and $\boldsymbol{p} \times \boldsymbol{q} = p_i \Theta^{ij} q_j/2$. The interaction is real, positive and translation invariant, but has an additional phase factor relative to the commutative theory. Due to momentum conservation the propagator in momentum representation only depends on the difference of the momenta $\Delta(\boldsymbol{k}, \boldsymbol{k}') = \delta^D(\boldsymbol{k} - \boldsymbol{k}')\Delta(\boldsymbol{k})$. As in the commutative case each contraction can thus be represented by an oriented line with definite momentum. The modified Feynman rules in momentum space are given by

$$= \frac{1}{k_i^2 + m^2}$$

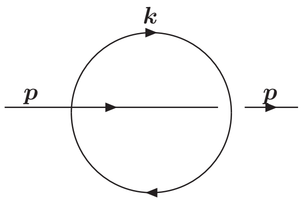


$$= \frac{\lambda}{4!}\ \mathrm{e}^{-\,\mathrm{i} \sum_{i<j=1}^{4} \boldsymbol{k}_i \times \boldsymbol{k}_j}$$

The additional mixing factor breaks the permutation symmetry of the lines at each vertex one is used to in the commutative case. The vertex is only invariant under cyclic permutations of the fields, which leads to two different kind of Feynman diagrams. Those which can be drawn on a sheet of paper without crossing of lines are called *planar diagrams*. Those which have crossed internal lines are called *non-planar diagrams*. Simple examples are given by the planar and non-planar tadpole:



Planar tadpole.



Non-planar tadpole.

Filk has shown [Fil96] that the vertex of a general Feynman diagram in this $\phi^{\star 4}$ theory can be simplified through the following two contractions

$$\hat{V}(\boldsymbol{k}_1, \ldots, \boldsymbol{k}_{n_1}, \boldsymbol{p})\hat{V}(\boldsymbol{k}_{n_1+1}, \ldots, \boldsymbol{k}_{n_2}, -\boldsymbol{p}) \quad = \quad (2\pi)^D \delta \left(\sum_{i=1}^{n_1} \boldsymbol{k}_i \right) \hat{V}(\boldsymbol{k}_1, \ldots, \boldsymbol{k}_{n_2}) \tag{2.48}$$

$$\hat{V}(\boldsymbol{k}_1, \ldots, \boldsymbol{k}_{n_1}, \boldsymbol{p}, \boldsymbol{k}_{n_1+1}, \ldots, \boldsymbol{k}_{n_2}, -\boldsymbol{p}) \quad = \quad \hat{V}(\boldsymbol{k}_1, \ldots, \boldsymbol{k}_{n_2}) \quad \text{for} \sum_{i=n_1+1}^{n_2} \boldsymbol{k}_i = 0\,. \tag{2.49}$$

The first of these *Filk moves* reduces a line by gluing together two vertices into bigger a one. Applying this move $n-1$ times to an n-vertex graph, one obtains a graph with all lines starting and ending at the same vertex, called a *rosette*. Planarity then describes the absence of crossing loop lines, for which the phase can be shown to cancel out using the second *Filk move*. Planar diagrams are thus identical to their commutative counterparts and have to be renormalized accordingly. Noncommutativity alone is thus not able to tame all UV divergences. However, the situation is even worse.

The non-planar diagrams carry additional phase factors coupling the internal and external lines. The initial hope that NCQFTs might be better behaved due to a natural UV cut-off however turned out to be too optimistic. Minwalla, Van Raamsdonk and Seiberg found an intriguing mixing of UV and IR degrees of freedom [MVRS00]. A famous example is the non-planar tadpole in 4 space dimensions which is given by

$$\frac{\lambda}{12}\int\frac{\mathrm{d}^4\boldsymbol{k}}{(2\pi)^4}\frac{-\,\mathrm{e}^{\,\mathrm{i}\,p_i\Theta^{ij}k_j}}{k_i^2+m^2}=\frac{\lambda}{48\pi^2}\sqrt{\frac{m^2}{(\Theta\cdot\boldsymbol{p})^2}}K_1(\sqrt{m^2(\Theta\cdot\boldsymbol{p})^2})\overset{p_i\to 0}{\sim}p_i^{-2}\,, \tag{2.50}$$

where K_1 is a modified Bessel function of the second kind. Contrary to the commutative case this diagram is finite for finite p due to the extra phase factor, however diverges as p_i^{-2} for $p_i\to 0$. A chain of these diagrams inserted into a bigger graph will inevitably lead to divergent integrals. A natural regularization in this plane wave basis is given by the restriction of the momenta to the annulus $|\Lambda_0|<|p|<|\Lambda|$. However, the oscillations imply that a UV cutoff Λ generates an effective IR cutoff $\Lambda_1=1/|\theta|\Lambda$, which is the root of the UV/IR mixing. This makes the Wilsonian renormalization impossible, since it would require a clear separation of high and low momentum scales. A general investigation of the renormalizability has been performed in [CR01]. Since divergences coming from non-planar diagrams cannot be absorbed by planar counterterms, renormalizable theories have to have finite non-planar diagrams.

2.2.3 NCQFT on Minkowski Spacetime and Unitarity

The transition to NCQFTs on Minkowski spacetime is formally straightforward. The classical action of the $\phi^{\star 4}$ theory in Minkowski spacetime reads

$$\mathcal{S}=\int_x\left(\frac{1}{2}\partial_\mu\phi\star\partial^\mu\phi-\frac{m^2}{2}\phi\star\phi-\frac{\lambda}{4!}\phi\star\phi\star\phi\star\phi\right)(\boldsymbol{x}) \tag{2.51}$$

while its quantum theory is formally given by

$$Z[J]=\mathcal{N}\int\mathcal{D}\phi\,\exp\left(\mathrm{i}\,\mathcal{S}+\int_x J(\boldsymbol{x})\phi(\boldsymbol{x})\right)\,. \tag{2.52}$$

with some normalization $\mathcal{N}$. The precise form of the path integral measure is not needed to determine $Z[J]$ perturbatively, since only the vanishing of the integrand for $|\phi|\to\infty$ is needed to find a differential equation for the generating functional. This is however not fulfilled, since the action is real and the integrand badly oscillating. This is usually remedied by adding for the time being the damping factor $\mathrm{i}\,\epsilon\int\phi^2$ to the action with $\epsilon>0$

$$Z[J]=\lim_{\epsilon\to 0^+}\mathcal{N}\int D\phi\,\exp\left(\mathrm{i}\,\mathcal{S}-\epsilon\int\phi^2+\int_x J(\boldsymbol{x})\phi(\boldsymbol{x})\right)\,, \tag{2.53}$$

which at the same time regularizes the singularity of the free propagator $(\partial_\mu^2+m^2)^{-1}$. Analoguesly to (2.35) we can derive for the free part of the generating functional $Z_0[J]=Z[J]_{\lambda=0}$

$$\lim_{\epsilon\to 0^+}(\partial_\mu^2+m^2-2\,\mathrm{i}\,\epsilon)\frac{\delta Z_0[J]}{\delta J(\boldsymbol{x})}=-\,\mathrm{i}\,J(\boldsymbol{x})Z_0[J]\,, \tag{2.54}$$

which has the solution

$$Z_0[J]=\mathcal{N}\exp\left(\frac{\mathrm{i}}{2}\int_x\int_y J(\boldsymbol{x})\,\Delta_F(\boldsymbol{x}-\boldsymbol{y})\,J(\boldsymbol{y})\right) \tag{2.55}$$

with Δ_F the Feynman propagator

$$\Delta_F(\boldsymbol{x})=\lim_{\epsilon\to 0^+}\int\frac{\mathrm{d}^D\boldsymbol{k}}{(2\pi)^D}\frac{\mathrm{e}^{\,\mathrm{i}\,k_\mu x^\mu}}{k_\mu^2-m^2+\mathrm{i}\,\epsilon}\,. \tag{2.56}$$

Using identical arguments as in section 2.2.1 the full generating functional is given by

$$Z[J] = \mathcal{N} \exp\left(\mathrm{i}\, \mathcal{S}_{\mathsf{int}} \left[\frac{\delta}{\delta J} \right] \right) Z_0[J] \tag{2.57}$$

leading to a perturbative expansion in terms of Feynman diagrams corresponding to $\mathcal{S}_{\mathsf{int}}$ and the usual Feynman propagators.

However, as has been found in [GM00], this perturbative setting leads to a violation of unitarity if space and *time* do not commute. The authors of [GM00] showed, that the cutting rule for the $\phi^{\star 3}$ two-point function and for the $\phi^{\star 4}$ four-point function are not fulfilled at one-loop order. As a necessary condition for a unitary S-matrix they found the positive definiteness of the expression

$$- p_\mu \Theta^{\mu\nu} \Theta_{\nu\sigma} p^\sigma \,, \tag{2.58}$$

which is not fulfilled for time/space noncommutative theories. In this case the analytical continuation of Euclidean Feynman diagrams produces new branch cuts that are responsible for the failure of the cutting rules.

This seems to contradict the common knowledge that a Hermitian interaction Hamiltonian H_I leads to a unitary S-matrix. And indeed, this remains true in the time/space noncommutative case [Bah04]. But, the Lagrangian formulation of the quantum theory in terms of the path integral is no longer equivalent to the Hamiltonian approach using the Dyson series and the interaction Hamiltonian H_I. As was pointed out in [BDFP02] the usual Wick theorem does not apply to non-local interactions. The contributions to the n-point function are given by

$$G_k(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n) = \frac{(-1)^n}{n!} \langle 0 | T \phi(\boldsymbol{x}_1) \cdots \phi(\boldsymbol{x}_n) H_I(t_1) \cdots H_I(t_k) | 0 \rangle \tag{2.59}$$

where T denotes the time ordering with respect to the time variables $x_1^0, \ldots, x_n^0$ and $t_1, \ldots, t_k$. The Wick theorem now tells us that all two-point functions $\Delta_+(\boldsymbol{x}) = \langle 0 | \phi(\boldsymbol{x}) \phi(0) | 0 \rangle$ and Heaviside step functions $\theta(x^0)$ coming from the time ordering can be combined to give a Feynman diagram in terms of Feynman propagators

$$\Delta_F(\boldsymbol{x}) = \theta(x_0) \Delta_+(\boldsymbol{x}) + \theta(-x_0) \Delta_+(-\boldsymbol{x}) \,. \tag{2.60}$$

This is not true for time/space-noncommutativity. The $\phi^{\star n}$ interaction Hamiltonian has the general form

$$H_I(t) = \int \prod_{i=1}^{n} \mathrm{d}^4 a_i \, G_t(a_1, \ldots, a_n) : \phi(a_1) \cdots \phi(a_n) : \,. \tag{2.61}$$

In this case the time ordering is with respect to the time variable t, called *interaction point*, and has no relation to the a_i at all. The perturbative analysis based on this "true" time ordering is known as *interaction point time ordering prescription* [LS02b, LS02c, B$^+$03]. The Heaviside functions in the Feynman propagator, however, come from an ordering of the "time" coordinates of the fields. Thus with time-space noncommuting coordinates the star-product and the time ordering no longer commute as is clearly visible from

$$\theta(x^0) \Delta_+^{\star 2}(\boldsymbol{x}) + \theta(-x^0) \Delta_+^{\star 2}(-\boldsymbol{x}) \neq \Delta_F^{\star 2}(\boldsymbol{x}) \tag{2.62}$$

due to

$$\theta \star \theta \neq \theta \,. \tag{2.63}$$

Actually, as has been pointed out in [DS03, Pia04], the Wick reduction does lead to the usual Feynman diagrams also for non-local theories, however with propagators given by

$$\mathcal{D}(\boldsymbol{x}, \tau) = \frac{1}{\mathrm{i}} \left(\theta(\tau) \Delta_+(\boldsymbol{x}) + \theta(-\tau) \Delta_+(-\boldsymbol{x}) \right) \tag{2.64}$$

where τ depends on the *interaction points* of the $H_I(t)$s. The star-products are performed with respect to the $\boldsymbol{x}$ occurring in two-point functions. For local interactions we find $\tau \equiv t$.

There are approaches towards a formulation of unitary NCQFTs in Minkowski spacetime with time-like noncommutativity. For models build on the Hamiltonian approach see e.g. [DFR95, Bah04] and [Pia10] for a nice review. UV/IR mixing is absent in this framework to lowest orders. Since perturbation theory gets complicated already at lower loops it is not clear whether it is completely free of UV/IR-mixing and might still be present in this framework. It has the disadvantage, or advantage, that different ways to define the interaction Hamiltonian are possible. A drawback is that the free fields do not obey the field equation even at tree level leading to a violation of current conservation. Yet another perturbative ansatz which is equivalent to the others on commutative, but not on noncommutative spacetime is the Yang-Feldman equation [BDFP02, Bah04].

The perturbative setup in the Hamiltonian approach is quite complicated such that it would be desirable to have an equivalent Euclidean path integral setup simplifying the combinatorial aspect of perturbation theory. The question is, what kind of Euclidean theory arises from a given Minkowskian theory and vice versa. In [Bah09] it has been shown that the Euclidean counterparts of the n-point functions for the Klein-Gordon theory on noncommutative Minkowski spacetime are not those following from the standard Euclidean setting, but appear with *on-shell* twisting factors, that is involving only on-shell momenta $p_\mu = (\omega_{\boldsymbol{p}}, p_a)$ for $a = 1, \ldots, D-1$ and $\omega_{\boldsymbol{p}} = \sqrt{p_a^2 + m^2}$.

We are interested in the other direction, starting with a Euclidean theory in a path integral setup. We will show that there exist models which allow for well-defined analytically continuations to Minkowski spacetime with help of a special regularization. These models are the LS-covariant models such as the Grosse-Wulkenhaar model and LSZ model, which at the same time have no UV/IR-mixing problem and are renormalizable to all orders in perturbation theory in Euclidean space. We are interested in the renormalization properties of their Minkowskian counterparts and the question, whether the unitarity problem still persists and in if yes in which sense. In the next chapter we will give a brief introduction to the LS-covariant models in Euclidean space and explain, how they are able to circumvent the UV/IR-mixing problem.

3 How to cure the UV/IR Mixing Problem

The UV/IR mixing poses severe problems to the renormalization program of NCQFTs. As was pointed out in [GW05b],

> *the message of the UV/IR entanglement is that noncommutativity relevant at short distances modifies the physics of the model at very large distances.*

The question is how to modify the theory? Nowadays there are two different approaches on the market which give an answer to this question, both defined on Euclidean space. The *LS-covariant models* are defined in section 3.2. We will demonstrate their covariance under the *Langmann-Szabo duality* (LS-duality) in section 3.3, which is seen to be responsible for their renormalizability and vanishing of their β-functions. In section 3.4 we will give a brief overview of the results which have been achieved in the last seven years. As an alternative to the LS-covariant models we briefly discuss another renormalizable model based on a different approach to cure the UV/IR mixing problem in section 3.5.

3.1 UV and IR Behavior of NCQFTs

The UV/IR mixing can be traced back to the non-locality of the theory. Let f and g be two fields, which are located in a small region $\Delta \ll \sqrt{\theta}$. Then one can show that contrary the star product of both is non-zero over a large region of size θ/Δ. As an extreme example one can take two delta functions, whose star product is constant throughout space

$$\delta(x) \star \delta(x) = \frac{1}{\det(\pi\theta)} \,. \tag{3.1}$$

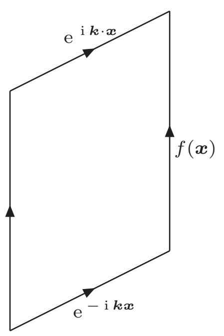
This shows that the interaction of noncommutative field theory is mediated by non-local extended objects instead of the point-like particles of ordinary quantum field theory. By exponentiation of the infinitesimal translations given by (2.23) to global translations we find

$$\mathrm{e}^{-\mathrm{i}\, k_\mu x^\mu} \star f(\boldsymbol{x}) \star \mathrm{e}^{\mathrm{i}\, k_\mu x^\mu} = f(\boldsymbol{x} + \Theta \cdot \boldsymbol{k}) \,. \tag{3.2}$$

One is thus tempted to imagine that a plane wave does not correspond to a particle, but to a "dipole", whose length is proportional to its transverse momentum [SJ99, BS00, DN01, Rey02]. For a dipole of momentum $\boldsymbol{k}$, its dipole moment is $\Theta \cdot \boldsymbol{k}$ and the position coordinate of the scalar field is Bopp shifted to the commutative coordinate

$$\boldsymbol{r} = \boldsymbol{x} + \Theta \cdot \boldsymbol{k} \,. \tag{3.3}$$

The ultraviolet dynamics in the regime $E \gg \Theta^{-1/2}$ are mediated through interactions of these dipoles who interact by joining at their ends:



Since the length of the dipoles is given by $|\Theta \cdot \boldsymbol{k}|$, a sharp ultraviolet cutoff Λ in momentum space induces an infrared cutoff at $1/(|\Theta|\Lambda)$, the inverse of the maximal dipole length.

On the other hand, the infrared dynamics in the regime $E \ll \Theta^{-1/2}$, where noncommutativity is negligible, are governed by the elementary quantum fields $\phi_{\boldsymbol{k}}$, which create pointlike quanta of momentum $\boldsymbol{k}$. This suggests that the UV/IR mixing problem may be understood as a mismatch between the dressed coordinates (3.3) and the elementary momenta $\boldsymbol{k}$, thus by the asymmetry between extended and pointlike degrees of freedom governing the different regimes. In order to cure this mismatch one can make the UV and IR regime symmetric via substitution of the generalized momenta

$$\boldsymbol{k} \longrightarrow \boldsymbol{k} + \boldsymbol{B} \cdot \boldsymbol{x}\,, \tag{3.4}$$

where the real constant $D \times D$ antisymmetric matrix $\boldsymbol{B}$ can be interpreted as an *electromagnetic background*. In terms of field theory, the natural implementation of this symmetrization is the replacement of usual derivatives by covariant derivatives

$$\partial_i \longrightarrow \partial_i + \mathrm{i}\, B_{ij} x^j\,. \tag{3.5}$$

with (B_{ij}) a $D \times D$ real, non-degenerated antisymmetric matrix. This is a generalization of the covariant derivative introduced by Filk [Fil90], as was illustrated at the end of section 2.1 where $B = (\Theta/2)^{-1}$. Contrary to (2.30), the free part now describes a Klein-Gordon field moving in a constant magnetic field perpendicular to the plane. Filk's action has not attracted any attention for more than ten years, until it turned out to be the crucial ingredient to successfully improve the renormalization properties of noncommutative quantum field theories. The various motivations and mathematical interpretations for the background field are summarized in [dG10].

3.2 LS-Covariant Models in Euclidean Space

Variations of the ansatz introduced above are the LSZ model [LSZ03, LSZ04], the Grosse-Wulkenhaar model [GW03, GW05b] and the *vulcanized Gross-Neveu model* [VT07b], all of them defined in Euclidean space. The symmetry of the position and momentum degrees of freedom is known as LS-duality, and manifests itself in an invariance of the theory under Fourier transformation plus a special scaling [LS02a]. Rivasseau et al. proposed to call the procedure of making a theory covariant under LS-duality (3.4) *vulcanization* (see footnote 1). A proof that this symmetry holds at the classical level for Euclidean and Minkowskian signature will be given in section 3.3 below. In order to prove the quantum version of this duality, we have to distinguish both cases. This is because the wave operators under consideration will have different spectral properties depending on the metric. The proof that this is a duality at quantum level will be handed in after the introduction of the matrix basis in chapter 4. The extension to Minkowski spacetime will be done in chapter 5.

3.2.1 LSZ Model

The *general Langmann-Szabo-Zarembo model* (LSZ model) in $D = 2n$ dimensions is a complex $\phi^{\star 4}$ theory. It is defined by the action $\mathcal{S}_{\mathsf{LSZ}} = \mathcal{S}_0 + \mathcal{S}_{\mathsf{int}}$ with

$$\begin{aligned} \mathcal{S}_0 &= \int \mathrm{d}^D\boldsymbol{x}\, \phi^*(\boldsymbol{x}) \left(\sigma \mathsf{K}_i^2 + (1-\sigma)\tilde{\mathsf{K}}_i^2 + \mu^2\right) \phi(\boldsymbol{x}) \\ \mathcal{S}_{\mathsf{int}} &= g \int \mathrm{d}^D\boldsymbol{x}\, \left[\alpha\, (\phi^* \star \phi \star \phi^* \star \phi)(\boldsymbol{x}) + \beta\, (\phi^* \star \phi^* \star \phi \star \phi)(\boldsymbol{x})\right] \end{aligned}\,. \tag{3.6}$$

The parameters are restricted to $\sigma \in [0,1]$, $\alpha, \beta \in \mathbb{R}_+$ and $\mu^2 > 0$ is the mass parameter. The *generalized momenta* K_i and *generalized dual momenta* $\tilde{\mathsf{K}}_i$ are given by

$$\begin{aligned} \mathsf{K}_i &= -\mathrm{i}\,\partial_i + B_{ij}x^j \\ \tilde{\mathsf{K}}_i &= -\mathrm{i}\,\partial_i - B_{ij}x^j \end{aligned} \tag{3.7}$$

for $i = 1, \dots D$ and obey the commutation relations

$$[\mathsf{K}_i, \mathsf{K}_j] = 2\,\mathrm{i}\,B_{ij} \quad , \quad [\tilde{\mathsf{K}}_i, \tilde{\mathsf{K}}_j] = -2\,\mathrm{i}\,B_{ij} \,. \tag{3.8}$$

and $[\mathsf{K}_i, \tilde{\mathsf{K}}_j] = 0$. Each of them describes a system in a constant magnetic field with field strength $\mp 2B_{ij}$, respectively. The coordinate system will be chosen such that the $D \times D$ dimensional deformation matrix Θ takes the canonical skew-symmetric form

$$(\Theta^{ij}) = \begin{pmatrix} 0 & \theta_1 & & & \\ -\theta_1 & 0 & & 0 & \\ & & \ddots & & \\ & 0 & & 0 & \theta_{D/2} \\ & & & -\theta_{D/2} & 0 \end{pmatrix} \tag{3.9}$$

with $\theta_k > 0$ and $k = 1, \dots, D/2$. The electromagnetic field strength B is of the same form

$$(B_{ij}) = \begin{pmatrix} 0 & B_1 & & & \\ -B_1 & 0 & & 0 & \\ & & \ddots & & \\ & 0 & & 0 & B_{D/2} \\ & & & -B_{D/2} & 0 \end{pmatrix} \tag{3.10}$$

with $B_k > 0$ and $k = 1, \dots, D/2$ and $B_k = 2\Omega/\theta_\ell$ for all k and $0 < \Omega \leq 1$. This implies that the wave operator $\sigma \mathsf{K}_i^2 + (1-\sigma)\tilde{\mathsf{K}}_i^2$ of the LSZ model in $D = 2n$ dimensions breaks down to a sum of n parts with

$$\begin{aligned} \mathsf{K}_i^2 &= \sum_{k=1}^{n} (\mathsf{P}_i^2)_k \\ \tilde{\mathsf{K}}_i^2 &= \sum_{k=1}^{n} (\tilde{\mathsf{P}}_i^2)_k \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} (\mathsf{P}_i^2)_k &= -(\partial_{2k-1}^2 + \partial_{2k}^2) - 2\,\mathrm{i}\,B_k(x^{2k}\partial_{2k-1} - x^{2k-1}\partial_{2k}) + B_k^2(x_{2k-1}^2 + x_{2k}^2) \\ (\tilde{\mathsf{P}}_i^2)_k &= -(\partial_{2k-1}^2 + \partial_{2k}^2) + 2\,\mathrm{i}\,B_k(x^{2k}\partial_{2k-1} - x^{2k-1}\partial_{2k}) + B_k^2(x_{2k-1}^2 + x_{2k}^2)\,. \end{aligned} \tag{3.12}$$

In the next chapter we will be concerned with diagonalizing the free action. Since all operators (3.12) commute with each other, the problem reduces to finding the eigenfunctions of one pair of operators $(\mathsf{P}_i^2)_k$ and $(\tilde{\mathsf{P}}_i^2)_k$. The interaction part consists of two inequivalent, noncommutative quartic interactions weighted by real parameters α and β. The α-part is known as oriented interaction while the β-part is called unoriented interaction. Up to now, renormalizability has only been shown for the oriented part.

For generic σ the free part can be rewritten as

$$\mathcal{S}_0 = \int \mathrm{d}^D\boldsymbol{x}\, \phi^*(\boldsymbol{x}) \left(\mathsf{K}_i^2|_{B \to \tilde{B}} + \Omega^2 \tilde{x}_i^2 + \mu^2 \right) \phi(\boldsymbol{x}) \tag{3.13}$$

with $\tilde{B} = (2\sigma - 1)B = (2\sigma - 1)(B_{ij})$, $\tilde{x}_i = 2\Theta_{ij}^{-1}x^j$ and $\Omega = B_k\theta_k/2$. The free part describes a massive complex scalar field coupled to a constant magnetic background and in a confining electric potential proportional to $\Omega^2\tilde{x}_i^2$. By adjusting the parameter σ we can switch between purely magnetic background at $\sigma = 0, 1$ or mixed magnetic and electric background.

The quantum theory will be defined by the generating functional for the LSZ model

$$Z[J, J^*] = \mathcal{N} \int \mathcal{D}\phi\, \mathcal{D}\phi^*\, \exp\left(-\mathcal{S}_{\mathsf{LSZ}} + \int_x J^*(\boldsymbol{x})\, \phi(\boldsymbol{x}) + \int_x \phi^*(\boldsymbol{x})\, J(\boldsymbol{x}) \right) . \tag{3.14}$$

Compared to the usual $\phi^{\star 4}$ model investigated in section 2.2.1 there are the additional terms in the free part of the action (apart from the extra degrees of freedom due to having complex instead of real fields). The external background will be treated exactly by using the *dressed propagator* of the field moving in this background, which is known as *Furry picture* [Fur51]. This is done by defining the free part of the action through all terms depending quadratically on the fields. The corresponding "free" generating functional is then a solution of

$$\begin{aligned}(\sigma \mathsf{K}_i^2 + (1-\sigma)\tilde{\mathsf{K}}_i^2 + \mu^2)\frac{\delta Z_0[J,J^*]}{\delta J^*(\boldsymbol{x})} &= J(\boldsymbol{x}) Z_0[J,J^*]\\ (\sigma \mathsf{K}_i^2 + (1-\sigma)\tilde{\mathsf{K}}_i^2 + \mu^2)\frac{\delta Z_0[J,J^*]}{\delta J(\boldsymbol{x})} &= J^*(\boldsymbol{x}) Z_0[J,J^*]\,.\end{aligned} \tag{3.15}$$

given by

$$Z_0[J,J^*] = \exp\left(\int_x \int_y J^*(\boldsymbol{x})\Delta(\boldsymbol{x},\boldsymbol{y})J(\boldsymbol{y})\right)\,. \tag{3.16}$$

with dressed propagator $\Delta(\boldsymbol{x},\boldsymbol{x}')$ determined by

$$(\sigma \mathsf{K}_i^2 + (1-\sigma)\tilde{\mathsf{K}}_i^2 + \mu^2)\Delta(\boldsymbol{x},\boldsymbol{y}) = \delta^2(\boldsymbol{x}-\boldsymbol{y})\,. \tag{3.17}$$

It should be noted that translation invariance is broken. The momentum is thus not conserved and the propagator $\Delta(\boldsymbol{x},\boldsymbol{y})$, does not depend solely on the difference $\boldsymbol{x}-\boldsymbol{y}$.[1] This implies that contrary to the usual $\phi^{\star 4}$ theory investigated in section 2.2.2, planar diagrams will not be identical to its commutative counterparts. The full interacting quantum theory is defined through

$$Z[J,J^*] = \mathcal{N}\,\exp\left(-\mathcal{S}_{\mathsf{int}}\left[\frac{\delta}{\delta J^*},\frac{\delta}{\delta J}\right]\right) Z_0[J,J^*]\,, \tag{3.18}$$

leading to modified Feynman diagrams with dressed propagator $\Delta(\boldsymbol{x},\boldsymbol{y})$. As will be shown in 3.3 the classical action is covariant under LS-duality. The proof that this symmetry holds in the full quantum theory will be given in section 4.5.

3.2.2 The Grosse-Wulkenhaar Model

The Grosse-Wulkenhaar model (in the following GW model for short) is a special case of the LSZ model defined above for $\sigma = 1/2$ and real fields. Because of its distinguished role it played in the process of understanding renormalizable NCQFTs we will give a brief account on it. Compared to the usual Klein-Gordon field, the free part moves in a harmonic oscillator potential, which amounts to replacing the Laplace operator according to

$$\partial_i^2 \longrightarrow \partial_i^2 - \Omega^2\tilde{x}_i^2\,, \quad \tilde{x}_i = 2\Theta^{-1}_{ij}x^j\,, \tag{3.19}$$

with frequency $\Omega = B_k\theta_k/2$. The action in $D=2n$ dimensions is given by

$$\mathcal{S}_{\mathsf{GW}} = \int \mathrm{d}^D\boldsymbol{x}\,\frac{1}{2}\phi(\boldsymbol{x})\left(-\partial_i^2 + \Omega^2\tilde{x}_i^2 + \mu^2\right)\phi(\boldsymbol{x}) + \mathcal{S}_{\mathsf{int}} \tag{3.20}$$

with interaction term

$$\mathcal{S}_{\mathsf{int}} = g\int \mathrm{d}^D\boldsymbol{x}\,(\phi\star\phi\star\phi\star\phi)(\boldsymbol{x})\,. \tag{3.21}$$

Despite being a real model, it is still covariant under LS-duality, as will be shown below. The perturbative setting for the GW model is defined through the partition function

$$Z[J] = \mathcal{N}\int \mathcal{D}\phi\,\exp\left(-\mathcal{S}_{\mathsf{GW}} + \int_x J(\boldsymbol{x})\phi(\boldsymbol{x})\right) \tag{3.22}$$

with real fields $\phi(\boldsymbol{x})$. The only difference to the usual $\phi^{\star 4}$ model investigated in section 2.2.1 is the additional oscillator term in the free part of the action. In the Furry picture, the free generating functional is thus given by

$$Z_0[J] = \exp\left(\frac{1}{2}\int_x\int_y J(\boldsymbol{x})\Delta(\boldsymbol{x},\boldsymbol{y})J(\boldsymbol{y})\right)\,. \tag{3.23}$$

[1] For $\sigma = 1$ we have invariance under *magnetic translations*, which is the invariance of translations plus a suitable gauge transformation of the magnetic fields: $\phi(\boldsymbol{x}) \mapsto \mathrm{e}^{\,\mathrm{i}\,\boldsymbol{a}\cdot B\cdot\boldsymbol{x}}\phi(\boldsymbol{x}+\boldsymbol{a})$. The free propagator then is of the form $\Delta(\boldsymbol{x},\boldsymbol{y}) = \mathrm{e}^{-\,\mathrm{i}\,\boldsymbol{x}\cdot B\cdot\boldsymbol{y}}\bar{\Delta}(\boldsymbol{x}-\boldsymbol{y})$. The same is true for $\sigma = 0$ with $B \to -B$.

with dressed propagator $\Delta(\boldsymbol{x},\boldsymbol{x}')$ determined by

$$(-\partial_i^2+\Omega^2\tilde{x}_i^2+\mu^2)\Delta(\boldsymbol{x},\boldsymbol{y})=\delta^2(\boldsymbol{x}-\boldsymbol{y})\,. \tag{3.24}$$

The momentum is not conserved and the propagator $\Delta(\boldsymbol{x},\boldsymbol{y})$ thus depends on both variables $\boldsymbol{x}$ and $\boldsymbol{y}$ independently. The perturbative setting for the full interacting theory is given

$$Z[J]=\mathcal{N}\,\exp\left(-\mathcal{S}_{\mathsf{int}}\left[\frac{\delta}{\delta J}\right]\right)Z_0[J]\,, \tag{3.25}$$

leading to the modified Feynman diagrams for the $\phi^{\star 4}$ vertex with the dressed propagator $\Delta(\boldsymbol{x},\boldsymbol{y})$.

3.2.3 Vulcanized Gross-Neveu Model

As an example for a fermionic LS-covariant model we will shortly present the vulcanized Gross-Neveu model. The usual Gross-Neveu model is a quantum field theory of two-dimensional Dirac fermions coupled through $(\overline{\psi}\psi)^2$ interaction terms. The free part of the *vulcanized Gross-Neveu model* (vGN model) is the usual fermionic Gross-Neveu model which has been made LS-covariant according to the prescription explained above. The action of the noncommutative vulcanized version (with only one flavor) reads

$$\mathcal{S}_{\mathsf{GN}}=\int \mathrm{d}^2\boldsymbol{x}\,\overline{\psi}(\boldsymbol{x})\left(\not{\mathsf{P}}+\mu\right)\psi(\boldsymbol{x})+V_o+V_{no}\,, \tag{3.26}$$

with $\not{\mathsf{P}}=\gamma^i\mathsf{P}_i$ and γ^1,γ^2 constituting a two-dimensional representation of the Clifford algebra

$$\{\gamma^i,\gamma^j\}=2\delta^{ij}\,. \tag{3.27}$$

The interaction terms are divided into orientable V_o and non-orientable V_{no} terms, given by

$$\begin{aligned}
V_o \;=\;& \frac{\lambda_1}{4}\sum_{a,b}\int \mathrm{d}^{2n}\boldsymbol{x}\,\overline{\psi}\star\psi\star\overline{\psi}\star\psi(\boldsymbol{x}) && (3.28)\\
&+\frac{\lambda_2}{4}\sum_{a,b}\int \mathrm{d}^{2n}\boldsymbol{x}\,\overline{\psi}\star\gamma^i\psi\star\overline{\psi}\star\gamma_i\psi(\boldsymbol{x}) && (3.29)\\
&+\frac{\lambda_3}{4}\sum_{a,b}\int \mathrm{d}^{2n}\boldsymbol{x}\,\overline{\psi}\star\gamma_5\psi\star\overline{\psi}\star\gamma_5\psi(\boldsymbol{x}) && (3.30)
\end{aligned}$$

and

$$\begin{aligned}
V_{no} \;=\;& \frac{\lambda_4}{4}\sum_{a,b}\int \mathrm{d}^{2n}\boldsymbol{x}\,\psi\star\overline{\psi}\star\overline{\psi}\star\psi(\boldsymbol{x}) && (3.31)\\
&+\frac{\lambda_5}{4}\sum_{a,b}\int \mathrm{d}^{2n}\boldsymbol{x}\,\psi\star\gamma^i\overline{\psi}\star\overline{\psi}\star\gamma_i\psi(\boldsymbol{x}) && (3.32)\\
&+\frac{\lambda_6}{4}\sum_{a,b}\int \mathrm{d}^{2n}\boldsymbol{x}\,\psi\star\gamma_5\overline{\psi}\star\overline{\psi}\star\gamma_5\psi(\boldsymbol{x}) && (3.33)
\end{aligned}$$

where $\gamma^5=\mathrm{i}\,\gamma^0\gamma^1$. Since there is no renormalization proof in matrix representation available we will not further investigate this model in the forthcoming chapters.

3.3 Classical LS-Covariance

We will now introduce the LS-duality and show that the models introduced above are indeed LS-covariant at the classical level. This result was initially proven in [LS02a] for the Euclidean space. We will reproduce it at this point to show that the proof also holds for Minkowskian signature.

For the interaction term we will need the following lemma

Lemma 3.1. *The multiple star product of functions $f_k \in \mathcal{S}(\mathbb{R}^D)$ for $k = 1, \dots 4$ has the following momentum and position space representations*

$$\begin{aligned}\int \mathrm{d}^D\boldsymbol{x}\, (f_1 \star f_2 \star f_3 \star f_4)(\boldsymbol{x}) &= \prod_{a=1}^{4}\left(\int \frac{\mathrm{d}^D\boldsymbol{x}_a}{(2\pi)^{D/2}}\right) f(\boldsymbol{x}_1)\, f(\boldsymbol{x}_2)\, f(\boldsymbol{x}_3)\, f(\boldsymbol{x}_4)\, V(\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_3,\boldsymbol{x}_4) \\ &= \prod_{a=1}^{4}\left(\int \frac{\mathrm{d}^D\boldsymbol{k}_a}{(2\pi)^{D/2}}\right) \hat{f}(\boldsymbol{k}_1)\, \hat{f}(\boldsymbol{k}_2)\, \hat{f}(\boldsymbol{k}_3)\, \hat{f}(\boldsymbol{k}_4)\, \hat{V}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3,\boldsymbol{k}_4)\end{aligned} \tag{3.34}$$

with vertex functions given by

$$\begin{aligned}V(\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_3,\boldsymbol{x}_4) &= \frac{(2\pi)^D}{|\det(\Theta/2)|}\delta^D(\boldsymbol{x}_1-\boldsymbol{x}_2+\boldsymbol{x}_3-\boldsymbol{x}_4)\,\mathrm{e}^{-\mathrm{i}\,\boldsymbol{x}_1\wedge\boldsymbol{x}_2-\mathrm{i}\,\boldsymbol{x}_3\wedge\boldsymbol{x}_4} \\ \hat{V}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3,\boldsymbol{k}_4) &= (2\pi)^D\delta^D(\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3+\boldsymbol{k}_4)\,\mathrm{e}^{-\mathrm{i}\,\boldsymbol{k}_1\times\boldsymbol{k}_2-\mathrm{i}\,\boldsymbol{k}_3\times\boldsymbol{k}_4}\,.\end{aligned} \tag{3.35}$$

where $\boldsymbol{p}\times\boldsymbol{q} = 2^{-1}p_i\Theta^{ij}q_j$ and $\boldsymbol{p}\wedge\boldsymbol{q} = 2p_i(\Theta^{-1})^{ij}q_j$.

Proof: is given in appendix A.

The spacetime metric does not play any role in this proof, since only Fourier expansions and Gaussian integrals were needed. Changing from Euclidean to Minkowskian metric amounts to interchanging Euclidean and Minkowskian scalar products in the expressions above. We will need a simple variation of this lemma. Using relation $\widehat{f^*}(\boldsymbol{k}) = \hat{f}^*(-\boldsymbol{k})$ we find

$$\begin{aligned}\int \mathrm{d}^D\boldsymbol{x}\, (f_1^* \star f_2 \star f_3^* \star f_4)(\boldsymbol{x}) &= \prod_{a=1}^{4}\left(\int \frac{\mathrm{d}^D\boldsymbol{k}_a}{(2\pi)^{D/2}}\right) \hat{f}_1^*(\boldsymbol{k}_1)\, \hat{f}_2(\boldsymbol{k}_2)\, \hat{f}_3^*(\boldsymbol{k}_3)\, \hat{f}_4(\boldsymbol{k}_4)\, \hat{V}(-\boldsymbol{k}_1,\boldsymbol{k}_2,-\boldsymbol{k}_3,\boldsymbol{k}_4) \\ \int \mathrm{d}^D\boldsymbol{x}\, (f_1^* \star f_2^* \star f_3 \star f_4)(\boldsymbol{x}) &= \prod_{a=1}^{4}\left(\int \frac{\mathrm{d}^D\boldsymbol{k}_a}{(2\pi)^{D/2}}\right) \hat{f}_1^*(\boldsymbol{k}_1)\, \hat{f}_2^*(\boldsymbol{k}_2)\, \hat{f}_3(\boldsymbol{k}_3)\, \hat{f}_4(\boldsymbol{k}_4)\, \hat{V}(-\boldsymbol{k}_1,-\boldsymbol{k}_2,\boldsymbol{k}_3,\boldsymbol{k}_4)\,.\end{aligned} \tag{3.36}$$

Note that these results are in clear contrast to the commutative case, in which the position and momentum vertices are very different. There we have a local position-space interaction vertex $V(\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_3,\boldsymbol{x}_4) \propto \delta^D(\boldsymbol{x}_1-\boldsymbol{x}_2+\boldsymbol{x}_3-\boldsymbol{x}_4)\delta^D(\boldsymbol{x}_1-\boldsymbol{x}_2)\delta^D(\boldsymbol{x}_2-\boldsymbol{x}_3)$ and a non-local interaction vertex in momentum space $\hat{V}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3,\boldsymbol{k}_4) \propto \delta^D(\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3+\boldsymbol{k}_4)$. The noncommutative action possess a duality between the UV and IR regime. In contrast to the usual free scalar action this manifests itself in a symmetry of the whole LSZ action. In the following we will reproduce the proof given in [LS02a], in order to show that the duality holds irrespectively of the signature of the metric.

Lemma 3.2 (Classical duality)**.** *The general LSZ action*

$$\mathcal{S}_{\mathsf{LSZ}} = \mathcal{S}_0 + \mathcal{S}_{\mathsf{int}} \equiv \mathcal{S}_{\mathsf{LSZ}}[\phi; B, g, \Theta] \tag{3.37}$$

defined above obeys

$$\mathcal{S}_{\mathsf{LSZ}}[\phi; B, g, \Theta] = \mathcal{S}_{\mathsf{LSZ}}[\tilde{\phi}; B, \tilde{g}, \tilde{\Theta}]\,, \tag{3.38}$$

where

$$\tilde{\phi}(\boldsymbol{x}) = \sqrt{|\det B|}\,\hat{\phi}(B\cdot\boldsymbol{x})\,, \tag{3.39}$$

$\hat{\phi}(\boldsymbol{k})$ the Fourier transform of $\phi(\boldsymbol{x})$, and the scalar product may have Euclidean or Minkowskian signature. The transformed coupling parameters are

$$\tilde{\Theta} = -4\,B^{-1}\Theta^{-1}B^{-1}\,, \quad \tilde{g} = |\det(B\Theta/2)|^{-1}\,g\,. \tag{3.40}$$

Moreover, the transformation $(\phi; B, g, \Theta) \mapsto (\phi; B, \tilde{g}, \tilde{\Theta})$ is a duality of the field theory, i.e. it generates a cyclic group of order two.

Proof: For the following we define the derivatives $\hat{\partial}^j = \partial/\partial k_j$ and $\tilde{\partial}_i = \partial/\partial \tilde{k}^i = -B^{ij}\hat{\partial}_j$ with $\tilde{\boldsymbol{k}} = B^{-1}\cdot\boldsymbol{k}$. We start with the mass term. Using the Parseval relation we get

$$\begin{aligned}
\mu^2 \int \mathrm{d}^D\boldsymbol{x}\,\phi^*(\boldsymbol{x})\,\phi(\boldsymbol{x}) &= \mu^2 \int \mathrm{d}^D\boldsymbol{k}\,\hat{\phi}^*(\boldsymbol{k})\,\hat{\phi}(\boldsymbol{k}) \\
&= \mu^2 \int \mathrm{d}^D\tilde{\boldsymbol{k}}\,|\det B|\,\hat{\phi}^*(B\cdot\tilde{\boldsymbol{k}})\,\hat{\phi}(B\cdot\tilde{\boldsymbol{k}}) \\
&= \mu^2 \int \mathrm{d}^D\boldsymbol{x}\,\tilde{\phi}^*(\boldsymbol{x})\,\tilde{\phi}(\boldsymbol{x})\,. \qquad (3.41)
\end{aligned}$$

where in the last we renamed $\tilde{\boldsymbol{k}} = \boldsymbol{x}$. Furthermore we get

$$\widehat{\mathsf{P}_i\phi} = (k_i + \mathrm{i}\,B_{ij}\hat{\partial}^j)\hat{\phi}(\boldsymbol{k}) = (-\mathrm{i}\,\tilde{\partial}_i + B_{ij}\tilde{k}^j)\hat{\phi}(\boldsymbol{k})\,. \qquad (3.42)$$

Thus defining $Q_i = \mathrm{i}\,\tilde{\partial}_i - B_{ij}\tilde{k}^j$ we can proceed as before using again the Parseval relation

$$\begin{aligned}
\int \mathrm{d}^D\boldsymbol{x}\,(\mathsf{P}_i\phi)^\dagger(\boldsymbol{x})\,(\mathsf{P}^i\phi)(\boldsymbol{x}) &= \int \mathrm{d}^D\tilde{\boldsymbol{k}}\,|\det B|\,(Q_i\hat{\phi})^\dagger(B\cdot\tilde{\boldsymbol{k}})\,(Q^i\phi)(B\cdot\tilde{\boldsymbol{k}}) \\
&= \int \mathrm{d}^D\boldsymbol{x}\,(Q_i\tilde{\phi})^\dagger(\boldsymbol{x})\,(Q^i\tilde{\phi})(\boldsymbol{x}) \qquad (3.43)
\end{aligned}$$

which has the same form as before with $\tilde{\phi}$ substituted for ϕ. The same analysis holds for the part containing $\tilde{\mathsf{P}}_i$, which proves the duality for the free part. Surely, these considerations are independent of the particular choice of the metric.

The symmetry of the interaction term $\mathcal{S}_{\mathsf{int}}$ follows immediately from lemma 3.1 and relations (3.36). Up to the term $|\det(\Theta/2)|^{-1}$, they have the same form in momentum and position space but with $(\Theta/2)^{-1}$ substituted for $\Theta/2$. Changing $\hat{\phi}\to\tilde{\phi}$ and $\boldsymbol{k}\to\tilde{\boldsymbol{k}}$ this implies

$$\begin{aligned}
g &\to |\det(B\Theta/2)|^{-1}g\,, \\
\Theta &\to -4B^{-1}\Theta^{-1}B^{-1}\,,
\end{aligned} \qquad (3.44)$$

which finally proves the lemma. □

At the special points $\Theta = \pm 2B^{-1}$ the field theory is completely invariant under Fourier transformation (up to the sign of θ), and it is said to be self-dual. It is important to notice that everything we needed to prove this theorem were Fourier and Gaussian integrals. This implies that this classical duality holds for Euclidean as well as for Minkowskian metric.

The proof of the classical duality in the LSZ case is based on the fact that the Fourier transformed complex conjugated fields get momenta with flipped sign. For real fields this has to be ensured artificially by using the *cyclic Fourier transformation* instead of the usual Fourier transformation, defined by

$$\hat{\phi}(\boldsymbol{k}_a) := \int \frac{\mathrm{d}^D\boldsymbol{x}_a}{(2\pi)^{D/2}}\,\phi(\boldsymbol{x}_a)\,\mathrm{e}^{\,\mathrm{i}\,(-1)^a\boldsymbol{k}_a\cdot\boldsymbol{x}_a}\,, \qquad (3.45)$$

where $a = 1,2,3,4$ enumerates the momenta involved. It ensures that the sign of the momenta in the kinetic and the interaction term is the same as in the LSZ case such that the integrations can be done in the same way, proving the duality for the GW model. In the literature this duality is sometimes presented in the equivalent form

Lemma 3.3. *Under the exchange of position and momenta*

$$p_i \leftrightarrow \tilde{x}_i \quad , \quad \hat{\phi}(\boldsymbol{k}) \leftrightarrow \sqrt{|\det(\Theta/2)|}\phi(\boldsymbol{x}) \qquad (3.46)$$

with $\hat{\phi}$ the cyclic Fourier transformed *field, the Grosse-Wulkenhaar model given by the action $\mathcal{S}_{\mathsf{GW}}$ transforms as*

$$\mathcal{S}_{\mathsf{GW}}[\phi;\Omega,\lambda,\mu] = \Omega^2\mathcal{S}_{\mathsf{GW}}\left[\phi;\frac{1}{\Omega},\frac{\lambda}{\Omega^2},\frac{\mu}{\Omega}\right] \qquad (3.47)$$

At $\Omega = \pm 1$ the theory is again invariant under LS duality.

The interaction terms of the vGN model are identical to those of the scalar models given by the relations (3.36). Using

$$\widehat{\mathsf{P}_i\psi} = (k_i - \mathrm{i}\, B_{ij}\hat{\partial}^j)\hat{\psi}(\boldsymbol{k}) \tag{3.48}$$

it is clear that also the vGN model is covariant under Fourier transformation plus some appropriate rescaling of the fields, namely the LS-duality.

3.4 LS-Covariance, Renormalizability and Vanishing of the β-Function

The LS-duality covariance has been turned out to be a crucial concept in the construction of renormalizable noncommutative quantum field theories on Euclidean space. As a motivation for the search of corresponding theories on Minkowski spacetime, we will now give a brief overview on established results.

For the LSZ model there are two independent interaction terms. However, only the oriented interaction, i.e. the α-dependent part has been shown to be renormalizable. In the following β is always assumed to be 0. The behavior of these models strongly depend on their parameters σ and Ω. There are four cases which are generally distinguished:

- $\sigma = 1, \Omega = 1$ (critical and self-dual)
- $\sigma < 1, \Omega = 1$ (self-dual)
- $\sigma < 1, 0 < \Omega < 1$ (ordinary)
- $\sigma = 1, 0 < \Omega < 1$ (critical)

Each model may be complex or real. A model is called *critical* if the corresponding propagator in position space $\Delta(\boldsymbol{x}, \boldsymbol{x}')$ decays if $|\boldsymbol{x} - \boldsymbol{x}'|$ goes to infinity, but only oscillates as $|\boldsymbol{x} + \boldsymbol{x}'| \to \infty$.[2]

The critical and self-dual $\phi^{\star 4}$ model was first introduced in [Fil90], while its invariance under LS-duality has been pointed out [LS02a]. It has been shown to be exactly solvable [LSZ03, LSZ04] in general even dimensions, in the sense that there is a closed formula for the partition function for the regularized theory. The UV fixed point of the theory is, however, trivial, and the coupling constant vanishes if the UV cutoff is removed. The self-dual $\phi^{\star 3}$ theory in two, four and six dimensions, based on the real GW free action, has been shown to be renormalizable, non-trivial and essentially solvable genus by genus,[3] while in six dimensions this model is asymptotically free [GS06a, GS06b, GS08].

To improve the renormalization properties it was suggested in [LSZ03, LSZ04] to slightly disturb the LSZ model by choosing $\sigma < 1$. In this case the model gets altered by a harmonic oscillator term (see equation (3.13)), making the position-space propagator well behaved, with an exponential decay in $|\boldsymbol{x} - \boldsymbol{x}'| \to \infty$ and $|\boldsymbol{x} + \boldsymbol{x}'| \to \infty$ separately. The first result was due to Grosse and Wulkenhaar for $\sigma = 1/2$ and real fields. Using the matrix basis they showed that in two and four dimensions this theory is renormalizable to all orders in perturbation theory [GW03, GW05b]. While in two dimensions the harmonic oscillator frequency vanishes if the cutoff is removed and the theory is superrenormalizable, in four dimensions the selfdual point $\Omega = 1$ is a non-trivial fixed point of the theory. Their analysis relied on numerical determination of the scaling behavior of the propagator. This gap has later been filled by Rivasseau et al. [RVTW06], confirming the renormalizability. In addition, in four dimensions and at the self-dual point, the β-functions for both couplings Ω and g vanish to all orders in perturbation theory, and thus the renormalized couplings flows to a finite bare coupling [GW04, DR07, DGMR07]. This breakthrough has been possible due to certain Ward identities the model fulfills at quantum level, which are believed to be related to the LS duality. It is argued that the same is true for $\Omega < 1$, since the renormalization group flow of $\Omega \to 1$ is very fast [DGMR07]. These

[2] The designation *critical* is due to Rivasseau et al. [RT08]. In order to avoid confusion with "critical phenomena", it has been proposed to call them *covariant models* [Riv07b]. Since in this thesis we are already using the terminus *LS-covariant* models for all of these models we will stick to the description critical.

[3] Feynman diagrams in the perturbative expansion of NCQFTs form a Riemannian surface using the *double line formalism*, which we will introduce in section 4.4. The genus of a diagram is then identical to the genus of the surface.

results have been extended to all $0 < \sigma < 1$ for both real and complex fields in [GMRVT06, GGR09]. The extension to bosons with N flavors, called *color Grosse-Wulkenhaar model*, have been studied in [GR08]. They have been shown to be renormalizable and *asymptotic free* for $N > 1$.

The LSZ model for $\sigma = 1$ and $\Omega < 1$ is more difficult to treat. It belongs to the category of critical NCQFTs. It is shown to be renormalizable in 4 dimensions (see [RT08]). The vulcanized Gross-Neveu model is also of this type. In [VT07a] it has been shown that the massless orientable LS-duality covariant Gross-Neveu model is renormalizable to all orders in perturbation theory. Interestingly, the UV/IR mixing is partly still present, which, however, does not prevent the theory to be renormalizable. This seems to indicate that the precise role of LS-duality and the vulcanization procedure has not been fully understood yet. Furthermore it has been shown, that at one-loop level this theory is asymptotically "free" but not asymptotically safe [LVTW07], just like its commutative counterpart.

The scalar LS-covariant models have a vanishing β-function and thus contain no Landau ghost, contrary to the commutative ϕ^4_4 theory. Unlike non-abelian gauge theories, this elimination is achieved without asymptotic freedom, but instead with *asymptotic safety*. For these reasons, a full non-perturbative construction of the quantum field theory without any cut-off is believed possible [Riv07a, MR08], which would be the first known model in four dimensions. However, while the vanishing of the β-function was blessing from the constructive field theory point of view, it might turn out to be a problem from the physical perspective, as its connection to the commutative regime $\Theta \to 0$ may not exist. For this reason another renormalizable models has been suggested, called the *translation-invariant* model, briefly exposed in the next section.

3.5 Translation-Invariant Model

We will not keep quite about yet another concept which has successfully overcome the UV/IR mixing problem, but which at the same time avoids the breaking of translation invariance. It still keeps the UV/IR mixing under control and is renormalizable to all orders in perturbation theory [GMRT09]. It is called *translation-invariant model* and defined by the action

$$\mathcal{S}_{1/p^2} = \int \mathrm{d}^4\boldsymbol{x}\, \frac{1}{2}\left(\partial_i\phi(\boldsymbol{x})\,\partial^i\phi(\boldsymbol{x}) + \mu^2\phi^2(\boldsymbol{x}) - \phi(\boldsymbol{x}) \star \frac{a^2}{\theta^2\partial_i^2} \star \phi(\boldsymbol{x})\right) + \frac{\lambda}{4!}\int \mathrm{d}^4\boldsymbol{x}\, \phi^{\star 4}(\boldsymbol{x})\,, \tag{3.49}$$

with a a dimensionless constant and ∂_i^{-2} regarded as the Green function of ∂_i^2. The momentum space propagator, given by

$$G(k) = \frac{1}{k^2 + m^2 + \frac{a^2}{k^2}}\,, \tag{3.50}$$

does not affect the UV behavior, but has a nice damping in the IR regime. Putting n one-loop diagrams into one big loop has a nice IR behavior and thus solves the UV/IR mixing problem. It is renormalizable to all orders in perturbation theory [GMRT09].

The $1/p^2$-modified propagator can be seen as the usual propagator *dressed* by quantum corrections. Indeed, the $1/p^2$ corrections appear at every order in perturbation theory of the usual $\phi^{\star 4}$ theory. Its β-function is a rational multiple of the β-function of the commutative model [GT08]. It follows that contrary to the LS-covariant models it might not be realizable non-perturbatively, but it might have a meaningful commutative limit. In [MRT09] a commutative limit mechanism has been proposed, in which the $1/p^2$-terms get traded in for mass and wave function counterterms in the limit $\theta \to 0$. It is also argued that the extension to gauge theories is easier than in the LS-covariant models [GT08], since this extension preserves its trivial vacuum [BGK+08]. In this thesis we will not further follow this direction and restrict ourselves to the investigation of LS-covariant models. For more information see [Tan08, Tan10, BKSW10] and references therein.

4 Matrix Model Representation of Euclidean LS-Covariant NCQFTs

An invaluable tool in the investigation of the LS-covariant models has been the Landau basis, which allows to map them onto matrix models. It has been used to solve the critical, self-dual LSZ model exactly, to prove the renormalizability of the GW model and the vanishing of its β-functions. Though the renormalizability of these models have already been proven in position space, it will give us the possibility to define a well-defined analytical continuation to Minkowski spacetime.

From a physicist's point of view, the Landau basis is a very natural basis for the Hilbert space $L^2(\mathbb{R}^2)$, since it is made of (Wigner transformations of) the products of two copies of harmonic oscillator states. Furthermore, its elements are functions, which can be described as "best localized states" with respect to the star-product. The analog of Heisenberg's uncertainty relation for noncommuting coordinates forbids the simultaneous localization of conjugated coordinates x^i and x^j with $\theta^{ij} = \theta \neq 0$. If we try to localize Gaussian wave packets in two dimensions through a multiplication with itself one finds

$$\mathrm{e}^{-x_i^2/a^2} \star_\theta \mathrm{e}^{-x_i^2/a^2} = c\, \mathrm{e}^{-x_i^2/d^2} \tag{4.1}$$

with

$$c = 1 + \frac{\theta^2}{a^4} \quad \text{and} \quad d = \sqrt{\frac{a^4+\theta^2}{2a^2}}\,. \tag{4.2}$$

This implies that for $a > \sqrt{\theta}$ we get $d < a$ and for $a < \sqrt{\theta}$ we find $d > a$ with a fixed point at $a = \sqrt{\theta}$ and a best focused Gaussian given by

$$f_{00}(\boldsymbol{x}) \sim 2\,\mathrm{e}^{-x_i^2/\theta}\,. \tag{4.3}$$

The *Landau functions* $f_{mn} \in \mathcal{S}(\mathbb{R}^2)$ with $m, n \in \mathbb{N}$ are build on this Gaussian as a ground state via application of "ladder operators", and form an countable infinite orthonormal basis for $L^2(\mathbb{R}^2)$. They are energy eigenfunctions of the Landau Hamiltonian, which describes the motion of a charged particle, moving in a two-dimensional plane exposed to a perpendicular magnetic field, namely the $(\mathsf{P}_i^2)_k$ and $(\tilde{\mathsf{P}}_i^2)_k$. Expanding the fields in this basis, the theory gets mapped onto a matrix model, simplifying the interaction part considerably through the relation

$$f_{mn} \star f_{k\ell} \sim \delta_{nk} f_{m\ell}\,. \tag{4.4}$$

With help of this basis we get rid of the twisting factors showing up in the vertex functions in the plane wave basis, trading the noncommutative star-product in for the noncommutative matrix-product.

In the following we will derive the matrix model representation for Euclidean LS-covariant theories. This will be done in a fashion which will make it easy to capture the generalization to Minkowski spacetime. In section 4.1 we use the Weyl-Wigner transformation to map the eigenvalue equation for the operators $(\mathsf{P}_i^2)_k$ and $(\tilde{\mathsf{P}}_i^2)_k$ to the harmonic oscillator problem. Using its Fock space representation we construct the Landau functions in 4.2, which brings us immediately to the matrix model representation in section 4.3. Afterwards we will introduce the modified Feynman rules in terms of *ribbon graphs* and prove the LS-covariance at quantum level. In the following we will work in two dimensions. The generalization to higher dimensions is straightforward and will be given in section 4.6.

4.1 Mapping onto the Harmonic Oscillator

The Euclidean LS-covariant models introduced in the last chapter are special in the sense that there exists a matrix representation which diagonalizes the free part of the actions. This is implied by the fact that the operators $(\mathsf{P}_i^2)_k$ and $(\tilde{\mathsf{P}}_i^2)_k$ for each $k = 1, \ldots D/2$ have discrete spectra resembling the harmonic oscillator spectrum. In the following we will show how to see this and how to construct the eigenbasis.

We will skip the index k and work with the 2 dimensional wave operators P_i^2 and $\tilde{\mathsf{P}}_i^2$ acting on functions on $\mathcal{S}(\mathbb{R}^2)$ depending on $\boldsymbol{x} = (x, y)$. Since P_i^2 and $\tilde{\mathsf{P}}_i^2$ commute, we can find simultaneous eigenfunctions such that

$$\begin{aligned} \mathsf{P}_i^2\, f_{mn}(\boldsymbol{x}) &= \lambda_{mn} f_{mn}(\boldsymbol{x}) \\ \tilde{\mathsf{P}}_i^2\, f_{mn}(\boldsymbol{x}) &= \tilde{\lambda}_{mn} f_{mn}(\boldsymbol{x}) \end{aligned} \tag{4.5}$$

with $\lambda_{mn}, \tilde{\lambda}_{mn} \in \mathbb{R}$ and some indices m and n. The action of P_i^2 and $\tilde{\mathsf{P}}_i^2$ on some function $g(\boldsymbol{x})$ can be represented as

$$\begin{aligned} \mathsf{P}_i^2 g(\boldsymbol{x}) &= B^2(x^2 + y^2) \star_{2/B} g(\boldsymbol{x}) \\ \tilde{\mathsf{P}}_i^2 g(\boldsymbol{x}) &= g(\boldsymbol{x}) \star_{2/B} B^2(x^2 + y^2)\,, \end{aligned} \tag{4.6}$$

where $\star_{2/B}$ is the usual Moyal product with $\theta = 2/B$. This can be verified by using the perturbative representation of the star product to get

$$B^2(x^2 + y^2) \star_\theta g(\boldsymbol{x}) = \left(B^2(x^2 + y^2) + \mathrm{i}\, B^2\theta(x\partial_y - y\partial_x) - \frac{1}{4} B^2\theta^2(\partial_x^2 + \partial_y^2) \right) g(\boldsymbol{x})\,. \tag{4.7}$$

Setting $\theta = 2/B$ the rhs yields exactly $\mathsf{P}_i^2\, g(\boldsymbol{x})$ since

$$\mathsf{P}_i^2 = -\partial_i^2 - 2\,\mathrm{i}\, B(y\partial_x - x\partial_y) + B^2 x_i^2\,, \tag{4.8}$$

compare (3.12). Now interchanging the order of the two factors of the lhs of (4.7) flips the sign of θ on the rhs. This is equivalent to interchanging P_i^2 and $\tilde{\mathsf{P}}_i^2$.

The action of P_i^2 and $\tilde{\mathsf{P}}_i^2$ can thus be represented as a star product with the classical Hamiltonian $B^2(x^2 + y^2)$, which is the harmonic oscillator if we interpret y as the momentum conjugated to x. We can exploit this fact, by using the Weyl-Wigner transformation, which maps the star product of two functions onto the operator product of its Weyl symbols (2.14). The symbols

$$\hat{\mathcal{W}}\left[\sqrt{2}Bx\right] = \hat{\mathbf{q}} \quad \text{and} \quad \hat{\mathcal{W}}\left[\sqrt{2}By\right] = \hat{\mathbf{p}} \tag{4.9}$$

obey the Heisenberg algebra

$$[\hat{\mathbf{q}}, \hat{\mathbf{p}}] = 2B^2[x, y]_{\star_{2/B}} = \mathrm{i}\, 4B\,. \tag{4.10}$$

Noting that $\hat{\mathcal{W}}\left[x^2\right] = \hat{\mathcal{W}}\left[x\right]^2$ and $\hat{\mathcal{W}}\left[y^2\right] = \hat{\mathcal{W}}\left[y\right]^2$ we find

$$\begin{aligned} \mathsf{P}_i^2 f_{mn}(\boldsymbol{x}) &= \mathsf{W}\left[\hat{\mathbf{H}}_{\text{ho}}\hat{\mathbf{f}}_{mn}\right] = \lambda_{mn} f_{mn}(\boldsymbol{x}) \\ \tilde{\mathsf{P}}_i^2 f_{mn}(\boldsymbol{x}) &= \mathsf{W}\left[\hat{\mathbf{f}}_{mn}\hat{\mathbf{H}}_{\text{ho}}\right] = \tilde{\lambda}_{mn} f_{mn}(\boldsymbol{x}) \end{aligned} \tag{4.11}$$

with

$$\hat{\mathbf{H}}_{\text{ho}} = \frac{1}{2}(\hat{\mathbf{p}}^2 + \hat{\mathbf{q}}^2) \tag{4.12}$$

the harmonic oscillator and $\hat{\mathbf{f}}_{mn} = \hat{\mathcal{W}}\left[f_{mn}\right]$. Clearly, the left/right-eigenfunctions of $\hat{\mathbf{H}}_{\text{ho}}$ are tensor products of the form

$$\hat{\mathbf{f}}_{mn} = C_f |\phi_m\rangle\langle\phi_n| \tag{4.13}$$

with C_f being some constant and ϕ_n denoting the harmonic oscillator eigenstates. Working in the representation

$$\langle q'|\hat{\mathbf{q}}|q\rangle = \sqrt{2}Bq\,\langle q'|q\rangle \quad \Rightarrow \langle q'|\hat{\mathbf{p}}|q\rangle = -\,\mathrm{i}\,\frac{\partial}{\partial q/\sqrt{8}}\,\langle q'|q\rangle\,. \tag{4.14}$$

the harmonic oscillator is given by

$$\langle q'|\hat{\mathbf{H}}_{\text{ho}}|q\rangle \;=\; 4\left(-\partial_q + \gamma^2\, q^2\right)\langle q'|q\rangle \tag{4.15}$$

with $\gamma = B/2$. It is a self-adjoint operator on $L^2(\mathbb{R})$ with a discrete spectrum given by $\{8\gamma\,(n+1/2)\,,\, n \in \mathbb{N}\}$. Its eigenfunctions are known as *harmonic oscillator wavefunctions* and are given by

$$\phi_n(q) = \left(\frac{\sqrt{\gamma}}{2^n n! \sqrt{\pi}}\right)^{1/2} \mathrm{e}^{-\frac{\gamma}{2} q^2} H_n(\sqrt{\gamma}\, q)\,. \tag{4.16}$$

They form a Hilbert space basis for $L^2(\mathbb{R})$ and obey

$$\langle \phi_m | \phi_n \rangle = \delta_{mn}\,. \tag{4.17}$$

In summary, the simultaneous eigenvalue equations for P_i^2 and $\tilde{\mathsf{P}}_i^2$ are equivalent to two harmonic oscillator problems with eigenvalues given by

$$\lambda_{mn} = 4B\left(m + \frac{1}{2}\right) \quad \text{and} \quad \tilde{\lambda}_{mn} = 4B\left(n + \frac{1}{2}\right)\,, \tag{4.18}$$

and eigenfunctions being Wigner transformations of two harmonic oscillator states

$$f_{mn}^{(B)}(\boldsymbol{x}) = C_f \mathsf{W}\left[|\phi_m\rangle\langle\phi_n|\right](\boldsymbol{x}) \tag{4.19}$$

known as *Landau functions*. We use the superscript (B) to distinguish the Landau functions with different magnetic field strengths B.

4.2 Landau Functions

We will now construct the Landau functions and prove those properties which will be needed to find the matrix model representation of the LS-covariant models. In the following we will set $\theta = 2/B$, thus $\star = \star_{2/B}$. Using the explicit representation for the Wigner transformation (2.10) with $\theta = 2/B$ one gets

$$\begin{aligned} \int \mathrm{d}^2\boldsymbol{x}\, \mathsf{W}\left[|\phi_m\rangle\langle\phi_n|\right](\boldsymbol{x}) &= \int \mathrm{d}x \int \mathrm{d}y \int \mathrm{d}k\, \mathrm{e}^{\,\mathrm{i}\frac{B}{2}ky} \langle x + k/2 | \phi_m\rangle\langle\phi_n | x - k/2\rangle \\ &= \frac{4\pi}{B} \int \mathrm{d}x \langle x|\phi_m\rangle\langle\phi_n|x\rangle \\ &= \frac{4\pi}{B}\,. \end{aligned} \tag{4.20}$$

Thus demanding the normalization

$$\int \mathrm{d}^2\boldsymbol{x}\, f_{mn}^{(B)}(\boldsymbol{x}) = \sqrt{\frac{4\pi}{B}}\delta_{mn} \tag{4.21}$$

we find with (4.19)

$$f_{mn}^{(B)}(\boldsymbol{x}) = \sqrt{\frac{B}{4\pi}} \mathsf{W}\left[|\phi_m\rangle\langle\phi_n|\right](\boldsymbol{x})\,. \tag{4.22}$$

Using again the explicit expression for the Wigner transformation we can immediately deduce

$$f_{mn}^{(B)}(\boldsymbol{x})^* = \sqrt{\frac{B}{4\pi}} \int \mathrm{d}k\, \mathrm{e}^{-\mathrm{i}\frac{B}{2}ky} \langle x + k/2|\phi_n\rangle\langle\phi_m|x - k/2\rangle = f_{nm}^{(B)}(\boldsymbol{x})\,. \tag{4.23}$$

An important property can be proven by using the star-product relations for Wigner distributions (2.14) for the product of two Landau functions

$$\left(f_{mn}^{(B)} \star f_{k\ell}^{(B)}\right)(\boldsymbol{x}) = \frac{B}{4\pi}\mathsf{W}\left[|\phi_m\rangle\langle\phi_n|\phi_k\rangle\langle\phi_\ell|\right](\boldsymbol{x}) = \sqrt{\frac{B}{4\pi}}\delta_{nk} f_{m\ell}^{(B)}(\boldsymbol{x})\,. \tag{4.24}$$

which is called the *projector property* and allows us to map the noncommutative models to matrix models. Note that the definition of the Landau functions with field strength B implies the projector property only for $\star = \star_{2/B}$ thus for $\theta = 2/B$.[1] Combining the identities (4.21) and (4.24) we find that the $f^{(B)}_{mn}$ are orthonormal with respect to the L^2 scalar product

$$\begin{aligned}
\langle f^{(B)}_{mn} | f^{(B)}_{k\ell} \rangle &= \int \mathrm{d}\boldsymbol{x}\, f^{(B)}_{nm}(\boldsymbol{x})\, f^{(B)}_{k\ell}(\boldsymbol{x}) \\
&= \int \mathrm{d}\boldsymbol{x}\, (f^{(B)}_{nm} \star f^{(B)}_{k\ell})(\boldsymbol{x}) \\
&= \sqrt{\frac{B}{4\pi}} \int \mathrm{d}\boldsymbol{x}\, \delta_{mk}\, f^{(B)}_{n\ell}(\boldsymbol{x}) \\
&= \delta_{mn}\delta_{k\ell}\,.
\end{aligned} \tag{4.25}$$

The Landau functions have a simple form in terms of generalized Laguerre polynomials:

Lemma 4.1. *We define the radial coordinates $x = r\cos\varphi$ and $y = r\sin\varphi$. The Landau functions $f_{mn}(r,\varphi)$ are given by*

$$f^{(B)}_{mn}(r,\varphi) = (-1)^{\min(m,n)} \sqrt{\frac{B}{\pi}} \sqrt{\frac{\min(m!,n!)}{\max(m!,n!)}} \left(Br^2\right)^{|m-n|/2} \mathrm{e}^{\,\mathrm{i}\,\varphi(n-m)}\, \mathrm{e}^{\,-r^2/\theta} L^{|m-n|}_{\min(n,m)}\left(Br^2\right) \tag{4.26}$$

where $L^{\alpha}_{k}(x)$ are associated Laguerre polynomials.

A proof of this lemma will be given in section E.

The ladder operator representation of the harmonic oscillator states has an analog on the Wigner side, which will be useful to determine the matrix model representation of the NCQFTs. Observe that

$$\begin{aligned}
\mathsf{W}\left[|\phi_m\rangle\langle\phi_n|\right] &= \frac{1}{\sqrt{m!n!}} \mathsf{W}\left[(\hat{\mathbf{a}}^\dagger)^m |\phi_0\rangle\langle\phi_0| (\hat{\mathbf{a}})^n\right] \\
&= \frac{1}{\sqrt{m!n!}} (\mathsf{W}\left[\hat{\mathbf{a}}^\dagger\right])^{\star m} \star \mathsf{W}\left[|\phi_0\rangle\langle\phi_0|\right] \star (\mathsf{W}\left[\hat{\mathbf{a}}\right])^{\star n}\,.
\end{aligned} \tag{4.27}$$

The Fock space ladder operators for the harmonic oscillator with frequency $4B$ are defined by linear combinations $\hat{\mathbf{a}} = (\hat{\mathbf{q}} + \mathrm{i}\,\hat{\mathbf{p}})/\sqrt{8B}$. Using $\hat{\mathcal{W}}\left[\sqrt{2}Bx\right] = \hat{\mathbf{q}}$ and $\hat{\mathcal{W}}\left[\sqrt{2}By\right] = \hat{\mathbf{p}}$ we find for the Wigner transformed ladder operators

$$\mathsf{W}\left[\hat{\mathbf{a}}\right] = \sqrt{\frac{B}{4}}(x + \mathrm{i}\,y) \quad , \quad \mathsf{W}\left[\hat{\mathbf{a}}^\dagger\right] \sqrt{\frac{B}{4}}(x - \mathrm{i}\,y) \tag{4.28}$$

which are proportional to the complex coordinates $z = x + \mathrm{i}\,y$ and $\bar{z} = x - \mathrm{i}\,y$. We define new ladder operators $a, a^\dagger$ and $b, b^\dagger$ through

$$\begin{aligned}
\left(\sqrt{\frac{B}{4}}z\right) \star g(\boldsymbol{x}) = a\, g(\boldsymbol{x}) \quad &, \quad \left(\sqrt{\frac{B}{4}}\bar{z}\right) \star g(\boldsymbol{x}) = a^\dagger\, g(\boldsymbol{x}) \\
g(\boldsymbol{x}) \star \left(\sqrt{\frac{B}{4}}\bar{z}\right) = b\, g(\boldsymbol{x}) \quad &, \quad g(\boldsymbol{x}) \star \left(\sqrt{\frac{B}{4}}z\right) = b^\dagger\, g(\boldsymbol{x})\,.
\end{aligned} \tag{4.29}$$

Defining $\partial_z = \partial_x - \mathrm{i}\,\partial_y$ and $\partial_{\bar{z}} = \partial_x + \mathrm{i}\,\partial_y$ one notes that

$$\frac{\mathrm{i}}{B}(\partial_x\partial'_y - \partial_y\partial'_x) = \frac{1}{2B}(\partial_z\partial'_{\bar{z}} - \partial_{\bar{z}}\partial'_z)\,. \tag{4.30}$$

Using $\partial_z z = \partial_{\bar{z}}\bar{z} = 2$ one easily finds

$$\left(\sqrt{\frac{B}{4}}z\right) \star g(\boldsymbol{x}) = \left(\sqrt{\frac{B}{4}}z\right) g(\boldsymbol{x}) + 2\sqrt{\frac{B}{4}}\frac{1}{2B}\partial_{\bar{z}} g(\boldsymbol{x}) = \frac{1}{2}\left(\sqrt{B}z + \sqrt{\frac{1}{B}}\partial_{\bar{z}}\right) g(\boldsymbol{x}) \tag{4.31}$$

[1] We should remark that the f_{mn} also obey the projector property

$$\left(f^{(B)}_{mn} \star_{(-2/B)} f^{(B)}_{k\ell}\right)(\boldsymbol{x}) = \sqrt{\frac{B}{4\pi}}\delta_{m\ell} f^{(B)}_{kn}(\boldsymbol{x})\,.$$

which follows from $f \star_{-\theta} g = g \star_{\theta} f$.

and similarly

$$\begin{aligned} a &= \frac{1}{2}\left(\sqrt{\frac{1}{B}}\partial_{\bar{z}} + \sqrt{B}z\right) \quad , \quad a^\dagger = \frac{1}{2}\left(-\sqrt{\frac{1}{B}}\partial_z + \sqrt{B}\bar{z}\right) \\ b &= \frac{1}{2}\left(\sqrt{\frac{1}{B}}\partial_z + \sqrt{B}\bar{z}\right) \quad , \quad b^\dagger = \frac{1}{2}\left(-\sqrt{\frac{1}{B}}\partial_{\bar{z}} + \sqrt{B}z\right) . \end{aligned} \tag{4.32}$$

These operators indeed generate two commuting copies of the harmonic oscillator algebra

$$\begin{aligned} \left[a, a^\dagger\right] = \quad & \left[b, b^\dagger\right] = 1 \\ [a, b] = \quad & \left[a, b^\dagger\right] = 0 . \end{aligned} \tag{4.33}$$

A common vacuum state is defined by $af_{00}^{(B)} = bf_{00}^{(B)} = 0$. Demanding the normalization (4.21) one can use the explicit expressions (4.32) to solve for the ground state function

$$f_{00}^{(B)}(\boldsymbol{x}) = \sqrt{\frac{B}{\pi}}\, \mathrm{e}^{-\frac{B}{2}x_i^2} . \tag{4.34}$$

Applying the ladder operators we generate the Landau functions

$$f_{mn}^{(B)}(\boldsymbol{x}) = \frac{(a^\dagger)^m}{\sqrt{m!}}\frac{(b^\dagger)^n}{\sqrt{n!}} f_{00}^{(B)}(\boldsymbol{x}) . \tag{4.35}$$

and one easily finds

$$\begin{aligned} a\, f_{mn}^{(B)}(\boldsymbol{x}) &= \sqrt{m} f_{m-1,n}^{(B)}(\boldsymbol{x}) \quad , \quad a^\dagger\, f_{mn}^{(B)}(\boldsymbol{x}) = \sqrt{m+1}\, f_{m+1,n}^{(B)}(\boldsymbol{x}) \\ b\, f_{mn}^{(B)}(\boldsymbol{x}) &= \sqrt{n} f_{m,n-1}^{(B)}(\boldsymbol{x}) \quad , \quad b^\dagger\, f_{mn}^{(B)}(\boldsymbol{x}) = \sqrt{n+1}\, f_{m,n+1}^{(B)}(\boldsymbol{x}) . \end{aligned} \tag{4.36}$$

which will be important in the next section.

4.3 Matrix Model Representation

The Landau functions $f_{mn}^{(B)}$ fulfill the projector property with respect to the star-product $\star = \star_{2/B}$ thus with $\theta = 2/B$. However, the deformation parameters θ occurring in the interaction terms of the LS-covariant theories are not equal to $2/B$ in general. In this case we can either simplify the interaction *or* the free part of the action. Since we are able to find the matrix propagator even for $\theta \neq 2/B$, see section 8.2, we choose the first option and expand in $f_{mn}^{(2/\theta)}$, which obey the projector property with $\star = \star_\theta$ for $\theta \neq 2/B$.

For the following we assume the fields to be well-behaved functions which allow for an expansion in the Landau basis, like Schwartz functions (we will come back to this issue in section 6.2). From the action we can read off the perturbative definition of the correlation functions of the corresponding quantum theory. However, these will in general consist of products of distributions, and have to be regularized. An appropriate regularization is the matrix cutoff introduced in section 4.5. Removing this regularization is part of the renormalization program, and necessitates a good decay behavior of the propagator in the matrix representation for large indices (see section 8).

We start with the interaction part of the two-dimensional LSZ model given by[2]

$$\mathcal{S}_{\mathsf{int}} = g \int \mathrm{d}^2\boldsymbol{x}\, \left[\alpha\, (\phi^* \star \phi \star \phi^* \star \phi)(\boldsymbol{x}) + \beta\, (\phi^* \star \phi^* \star \phi \star \phi)(\boldsymbol{x})\right] \tag{4.37}$$

The scalar fields are expanded in Landau basis read

$$\begin{aligned} \phi(\boldsymbol{x}) &= \sum_{mn}^{\infty} f_{mn}^{(2/\theta)}(\boldsymbol{x})\, \phi_{mn} \\ \phi(\boldsymbol{x})^* &= \sum_{mn}^{\infty} f_{mn}^{(2/\theta)}(\boldsymbol{x})\, \overline{\phi}_{mn} \end{aligned} \tag{4.38}$$

[2]The generalization to higher dimensions is straightforward and given in section 4.6.

with coefficients given by

$$\begin{aligned} \phi_{mn} &= \langle f^{(2/\theta)}_{mn} | \phi \rangle = \int \mathrm{d}^2\boldsymbol{x}\, f^{(2/\theta)}_{nm}(\boldsymbol{x})\, \phi(\boldsymbol{x}) \\ \overline{\phi}_{mn} &= \langle f^{(2/\theta)}_{mn} | \overline{\phi} \rangle = \int \mathrm{d}^2\boldsymbol{x}\, f^{(2/\theta)}_{nm}(\boldsymbol{x})\, \phi(\boldsymbol{x})^* \,. \end{aligned} \tag{4.39}$$

Note that $\overline{\phi}_{mn} = \overline{\phi_{nm}} = (\phi_{nm})^* = (\phi_{mn})^\dagger$. Using

$$f^{(2/\theta)}_{m_1n_1} \star f^{(2/\theta)}_{m_2n_2} \star f^{(2/\theta)}_{m_3n_3} \star f^{(2/\theta)}_{m_4n_4} = \frac{1}{\sqrt{2\pi\theta}^3} \delta_{n_1m_2}\delta_{n_2m_3}\delta_{n_3m_4} f^{(2/\theta)}_{m_1n_4} \,, \tag{4.40}$$

the general $\phi^{\star 4}$ interaction term of the LSZ model simplifies to

$$\mathcal{S}_{\mathsf{int}} = \frac{g}{2\pi\theta} \sum_{n,m,k,\ell,p} \left(\alpha\, \overline{\phi}_{mn}\phi_{nk}\, \overline{\phi}_{k\ell}\phi_{\ell m} + \beta\, \overline{\phi}_{mn}\overline{\phi}_{nk}\, \phi_{k\ell}\phi_{\ell m} \right) \,. \tag{4.41}$$

For the Grosse-Wulkenhaar model this simply reads

$$\mathcal{S}_{\mathsf{int}} = \frac{g}{2\pi\theta} \sum_{mnk\ell} \phi_{mn}\phi_{nk}\phi_{k\ell}\phi_{\ell m} \tag{4.42}$$

We already know that at the *self-dual* points $\theta = \pm 2/B$ the Landau functions diagonalizes also the free part of the actions. The wave operator P^2 becomes

$$\begin{aligned} \mathsf{P}_i^2 &= 4B \left(a^\dagger a + \frac{1}{2} \right) \quad \text{for } \theta = +2/B \\ \mathsf{P}_i^2 &= 4B \left(b^\dagger b + \frac{1}{2} \right) \quad \text{for } \theta = -2/B \,, \end{aligned} \tag{4.43}$$

and analogously for $\tilde{\mathsf{P}}_i^2$ with $a, a^\dagger \leftrightarrow b, b^\dagger$ interchanged. Their matrix representations at $\theta = 2/B$ then simply read

$$\begin{aligned} (\mathsf{P}_i^2)_{mn;k\ell} &= \int_x f^{(2/\theta)}_{mn}(\boldsymbol{x})\, \mathsf{P}_i^2\, f^{(2/\theta)}_{k\ell}(\boldsymbol{x}) = 4B \left(k + \frac{1}{2} \right) \delta_{m\ell}\delta_{nk} \\ (\tilde{\mathsf{P}}_i^2)_{mn;k\ell} &= \int_x f^{(2/\theta)}_{mn}(\boldsymbol{x})\, \tilde{\mathsf{P}}_i^2\, f^{(2/\theta)}_{k\ell}(\boldsymbol{x}) = 4B \left(\ell + \frac{1}{2} \right) \delta_{m\ell}\delta_{nk} \,. \end{aligned} \tag{4.44}$$

For generic θ the expressions are more complicated:

Lemma 4.2. *The wave operator of the two-dimensional LSZ model in matrix representation is given by*

$$\begin{aligned} G_{mn;k\ell} \;=\; & \left(\mu^2 + 2\frac{(1+\Omega^2)}{\theta}(m+n+1)\delta_{m\ell}\delta_{nk} + \frac{4\tilde{\Omega}}{\theta}(n-m) \right) \delta_{m\ell}\delta_{nk} \\ & + 2\frac{\Omega^2 - 1}{\theta} \left(\sqrt{nm}\, \delta_{m,\ell+1}\, \delta_{n,k+1} + \sqrt{(n+1)(m+1)}\, \delta_{m,\ell-1}\, \delta_{n,k-1} \right) \,, \end{aligned} \tag{4.45}$$

with frequencies $\Omega = B\theta/2$ and $\tilde{\Omega} = (2\sigma - 1)\Omega$

Proof: The wave operator in matrix representation is given by

$$G_{mn;k\ell} = \int_x f^{(2/\theta)}_{mn}(\boldsymbol{x}) \left(\sigma \mathsf{P}_i^2 + (1-\sigma)\tilde{\mathsf{P}}_i^2 + \mu^2 \right) f^{(2/\theta)}_{k\ell}(\boldsymbol{x}) \,. \tag{4.46}$$

We show the following relation:

$$\mathsf{P}_i^2 = \frac{1}{2\theta} \left[(2 + B\theta)^2 \left(a^\dagger a + \frac{1}{2} \right) + (2 - B\theta)^2 \left(b^\dagger b + \frac{1}{2} \right) + \left(\theta^2 B^2 - 4 \right) \left(a^\dagger b^\dagger + ab \right) \right] \,. \tag{4.47}$$

This can be verified by inserting the explicit expressions (4.32) for a and b with $2/\theta$ substituted for B. Noting that

$$
\begin{aligned}
a^\dagger b^\dagger + ab &= \frac{1}{2}\left(\frac{2}{\theta}z\bar{z} + \frac{\theta}{2}\partial_z\partial_{\bar{z}}\right) \\
\left(a^\dagger a + \frac{1}{2}\right) + \left(b^\dagger b + \frac{1}{2}\right) &= \frac{1}{2}\left(\frac{2}{\theta}z\bar{z} - \frac{\theta}{2}\partial_z\partial_{\bar{z}}\right) \\
\left(a^\dagger a + \frac{1}{2}\right) - \left(b^\dagger b + \frac{1}{2}\right) &= \frac{1}{2}\left(\bar{z}\partial_{\bar{z}} - z\partial_z\right) .
\end{aligned}
\tag{4.48}
$$

we indeed find

$$
\begin{aligned}
\mathsf{P}_i^2 &= \frac{1}{2\theta}\left[(4+B^2\theta^2)\frac{1}{2}\left(\frac{2}{\theta}z\bar{z} - \frac{\theta}{2}\partial_z\partial_{\bar{z}}\right) + (B^2\theta^2-4)\frac{1}{2}\left(\frac{2}{\theta}z\bar{z} + \frac{\theta}{2}\partial_z\partial_{\bar{z}}\right) + 4B\theta\frac{1}{2}\left(\bar{z}\partial_{\bar{z}} - z\partial_z\right)\right] \\
&= \frac{1}{2\theta}\left[B^2 2\theta z\bar{z} - 2\theta\partial_z\partial_{\bar{z}} + 4B\theta\frac{1}{2}\left(\bar{z}\partial_{\bar{z}} - z\partial_z\right)\right] \\
&= B^2x_i^2 - \partial_i^2 + 2\,\mathrm{i}\,B(x\partial_y - y\partial_x) \\
&= (-\,\mathrm{i}\,\partial_i + B_{ij}x^j)_i^2 .
\end{aligned}
\tag{4.49}
$$

The corresponding expression for $\tilde{\mathsf{P}}_i^2$ are obtained by interchanging $a, a^\dagger \leftrightarrow b, b^\dagger$. Using (4.36) one can easily read off the matrix versions of the wave operators P_i^2 and $\tilde{\mathsf{P}}_i^2$

$$
\begin{aligned}
(\mathsf{P}_i^2)_{mn;k\ell} &= \frac{1}{2\theta}\Bigg[(2+B\theta)^2\left(n+\frac{1}{2}\right)\delta_{m\ell}\,\delta_{nk} + (2-B\theta)^2\left(m+\frac{1}{2}\right)\delta_{m\ell}\,\delta_{nk} \\
&\qquad + (\theta^2B^2-4)\left(\sqrt{nm}\,\delta_{m,\ell+1}\,\delta_{n,l+1} + \sqrt{(n+1)(m+1)}\,\delta_{m,\ell-1}\,\delta_{n,k-1}\right)\Bigg]
\end{aligned}
\tag{4.50}
$$

and

$$
\begin{aligned}
(\tilde{\mathsf{P}}_i^2)_{mn;k\ell} &= \frac{1}{2\theta}\Bigg[(2+B\theta)^2\left(m+\frac{1}{2}\right)\delta_{m\ell}\,\delta_{nk} + (2-B\theta)^2\left(n+\frac{1}{2}\right)\delta_{m\ell}\,\delta_{nk} \\
&\qquad + (\theta^2B^2-4)\left(\sqrt{nm}\,\delta_{m,\ell+1}\,\delta_{n,k+1} + \sqrt{(n+1)(m+1)}\,\delta_{m,\ell-1}\,\delta_{n,k-1}\right)\Bigg] .
\end{aligned}
\tag{4.51}
$$

These can be combined to give

$$
\begin{aligned}
&\sigma(\mathsf{P}_i^2)_{mn;k\ell} + (1-\sigma)(\tilde{\mathsf{P}}_i^2)_{mn;k\ell} \\
&= \frac{1}{2\theta}\sigma\left[(2+B\theta)^2\left(n+\frac{1}{2}\right)\delta_{m\ell}\,\delta_{nk} + (2-B\theta)^2\left(m+\frac{1}{2}\right)\delta_{m\ell}\,\delta_{nk}\right] \\
&\quad + \frac{1}{2\theta}(1-\sigma)\left[(2+B\theta)^2\left(m+\frac{1}{2}\right)\delta_{m\ell}\,\delta_{nk} + (2-B\theta)^2\left(n+\frac{1}{2}\right)\delta_{m\ell}\,\delta_{nk}\right] \\
&\quad + \frac{1}{2\theta}\left[(\theta^2B^2-4)\left(\sqrt{nm}\,\delta_{m,\ell+1}\,\delta_{n,k+1} + \sqrt{(n+1)(m+1)}\,\delta_{m,\ell-1}\,\delta_{n,k-1}\right)\right] \\
&= \frac{2}{\theta}\Big[(1+(B\theta/2)^2)(m+n+1)\delta_{m\ell}\delta_{nk} + (2\sigma-1)B\theta(n-m)\delta_{m\ell}\delta_{nk} \\
&\quad + ((\theta B/2)^2-1)\left(\sqrt{nm}\,\delta_{m,\ell+1}\,\delta_{n,k+1} + \sqrt{(n+1)(m+1)}\,\delta_{m,\ell-1}\,\delta_{n,k-1}\right)\Big] ,
\end{aligned}
\tag{4.52}
$$

which proves the theorem. □

The two-dimensional general LSZ model then has the matrix model representation

$$
\begin{aligned}
\mathcal{S}_{\mathsf{LSZ}} &= \sum_{mnk\ell} \overline{\phi}_{mn}G_{mn;k\ell}\phi_{\ell k} \\
&\quad + \frac{g}{2\pi\theta}\sum_{mnk\ell}\left(\alpha\,\overline{\phi}_{mn}\phi_{nk}\,\overline{\phi}_{k\ell}\phi_{\ell m} + \beta\,\overline{\phi}_{mn}\overline{\phi}_{nk}\phi_{k\ell}\,\phi_{\ell m}\right) .
\end{aligned}
\tag{4.53}
$$

Note that at $\sigma = 1$ the infinite Landau level degeneracy, i.e. the dependence on only one of the two Landau indices, manifests itself in a $U(\infty)$ symmetry

$$\phi \longrightarrow U \cdot \phi \quad , \quad \phi^\dagger \longrightarrow \phi^\dagger \cdot U^\dagger \, . \tag{4.54}$$

This is the maximal symmetry-group of area-preserving diffeomorphisms, and it acts through rotations of the magnetic quantum numbers n. The phase space becomes degenerate and the wave functions depend only on one half of the position space coordinates, leading to a reduction of the quantum Hilbert space at $\theta = 2/B$ [LSZ04]. In position space, this implies an oscillatory behavior of the propagator in the *long variable* $|\boldsymbol{x} + \boldsymbol{x}'| \to \infty$, making the renormalization procedure more involved [RT08].

From lemma 4.2 we can easily follow for $\sigma = 1/2$

Lemma 4.3. *The two-dimensional Grosse-Wulkenhaar wave operator in matrix representation is given by*

$$\begin{aligned} G_{mn;k\ell} \quad &= \quad \left(\mu^2 + 2\frac{\Omega^2+1}{\theta}(m+n+1)\right)\delta_{m\ell}\,\delta_{nk} \\ &\quad + 2\frac{\Omega^2-1}{\theta}\left(\sqrt{nm}\delta_{m,\ell+1}\,\delta_{n,k+1} + \sqrt{(n+1)(m+1)}\delta_{m,\ell-1}\delta_{n,k-1}\right) \end{aligned} \tag{4.55}$$

with frequency $\Omega = B\theta/2$.

The corresponding action reads

$$\mathcal{S}_{GW} = \sum_{mn;k\ell}\left(\frac{1}{2}\phi_{nm}G_{mn;k\ell}\phi_{k\ell} + \frac{g}{2\pi\theta}\phi_{mn}\phi_{nk}\phi_{k\ell}\phi_{\ell m}\right) . \tag{4.56}$$

4.4 Perturbative Quantum Field Theory in Matrix Representation

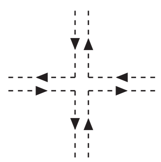
In the following we will demonstrate how the LS-covariant quantum field theories can be defined perturbatively in the matrix representation. The matrix representation of GW model is straightforwardly obtained using the perturbative solution of the generating functional (3.25) and the expansion of the GW action as in (4.56), which leads to the generating functional

$$Z[J] = \mathcal{N}\exp\left(-g\sum_{mnk\ell}\frac{\partial^4}{\partial J_{m\ell}\partial J_{\ell k}\partial J_{kn}\partial J_{nm}}\right)\exp\left(\frac{1}{2}\sum_{mnk\ell}J_{mn}\Delta_{mn;k\ell}J_{k\ell}\right) \tag{4.57}$$

The propagator solves the equation

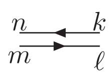
$$\sum_{k\ell} G_{mn;k\ell}\Delta_{\ell k;sr} = \sum_{k\ell}\Delta_{nm;\ell k}G_{k\ell;rs} = \delta_{mr}\delta_{ns} \, . \tag{4.58}$$

A precise expression for $\Delta_{mn;k\ell}$ will be determined in section 8.2.2. The modified Feynman rules are conveniently presented in the double line formalism. The vertex is diagonal and given by



$$= -g\,\delta_{mp}\delta_{nq}\delta_{kr}\delta_{\ell s} \, .$$

and the double line for the propagator



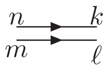
$$= \Delta_{nm;\ell k} \, .$$

Since the GW model considers real fields the Feynman diagrams are *unoriented*. The double lines have no distinguished direction, with arrows showing in both directions. They are usually kept for bookkeeping purposes.

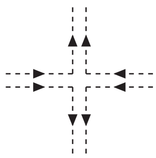
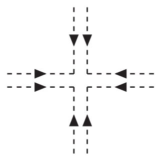
For complex fields each double line has arrows directed in the same direction, either incoming or outgoing. The matrix representation of the LSZ model reads

$$\begin{aligned} Z[J] = \mathcal{N} \exp &\left(-\alpha\, g \sum_{mnk\ell} \frac{\partial^4}{\partial J_{m\ell} \partial \overline{J}_{\ell k} \partial J_{kn} \partial \overline{J}_{nm}} - \beta\, g \sum_{mnk\ell} \frac{\partial^4}{\partial J_{m\ell} \partial J_{\ell k} \partial \overline{J}_{kn} \partial \overline{J}_{nm}} \right) \\ \times \quad &\exp \left(\sum_{mnk\ell} \overline{J}_{mn} \Delta_{mn;k\ell} J_{k\ell} \right) , \end{aligned} \tag{4.59}$$

The double lines are now oriented from ϕ^* to ϕ :

 $= \Delta_{nm;\ell k} \quad ,$

The two interaction terms $\phi^* \star \phi \star \phi^* \star \phi$ and $\phi^* \star \phi^* \star \phi \star \phi$ are represented by different diagrams

 $\sim \phi^* \star \phi \star \phi^* \star \phi$  $\sim \phi^* \star \phi^* \star \phi \star \phi$

having vertices $-g\, \delta_{mp} \delta_{nq} \delta_{kr} \delta_{\ell s}$ times α or β. Restricting to one of these interactions reduces the number possible diagrams for the complex matrix model.

Every Feynman diagram is represented by a *ribbon graph*. Its topological data is now decisive for the question whether it is divergent or not. The *power counting theorem* for general *non-local* matrix models has been proven by Grosse and Wulkenhaar in [GW05a].[3] More on this in chapter 8

4.5 LS-Duality at Quantum Level

To ensure the LS-duality at the quantum level, we have to find a regularization scheme for the model, which suppresses possible divergences and at the same time keeps the duality manifestly. We demonstrate the procedure at the GW model. Using this regularization scheme, Grosse and Wulkenhaar were able to prove the renormalizability of the GW model in two and four dimensions [GW03, GW05b].

Connected Green functions with M external legs are given by

$$\mathcal{G}_M(\boldsymbol{x}_1, \dots, \boldsymbol{x}_M; B, g, \Theta) = \prod_{i=1}^{M} \frac{\delta}{\delta J(\boldsymbol{x}_i)} W(J; B, g, \Theta) \bigg|_{J=0} . \tag{4.60}$$

with

$$W[J] = -\ln \frac{Z[J]}{Z[0]} \equiv W[J; B, g, \Theta] \,. \tag{4.61}$$

the generating functional of connected graphs. Since the path integral measure is formally invariant under $\phi \to \tilde{\phi}$, the duality symmetry of the classical action plus the identity

$$\int_x \phi(\boldsymbol{x})\, J(\boldsymbol{x}) = \int_x \tilde{\phi}(\boldsymbol{x})\, \tilde{J}(\boldsymbol{x}) \tag{4.62}$$

[3] A matrix model is thereby called *local* if $\Delta_{nm;\ell k} = \Delta(m,n) \delta_{m\ell} \delta_{nk}$ for some function $\Delta(m,n)$ and *non-local* otherwise.

formally yields the identity

$$W[J;B,g,\Theta]=W[\tilde{J};B,\tilde{g},\tilde{\Theta}]\,. \tag{4.63}$$

Hence any connected Green function with M external legs formally obeys the identity

$$\hat{\mathcal{G}}_M(\boldsymbol{k}_1,\dots,\boldsymbol{k}_M;B,g,\Theta)=|\det(B)|^{M/2}\,\mathcal{G}_M\left(\tilde{\boldsymbol{k}}_1,\dots,\tilde{\boldsymbol{k}}_M;B,\tilde{g},\tilde{\Theta}\right) \tag{4.64}$$

with $\hat{\mathcal{G}}_M$ the Fourier transform of $\mathcal{G}_M$ and as before $\tilde{\boldsymbol{k}}=\boldsymbol{B}^{-1}\cdot\boldsymbol{k}$.

However, to nail this symmetry down, one has to regularize possible divergences while keeping this duality manifestly at any step in perturbation theory. Note that the propagator for the two-dimensional GW model reads

$$\Delta(\boldsymbol{x},\boldsymbol{x}')=\sum_{m,n}\frac{f^{(B)}_{mn}(\boldsymbol{x})f^{(B)}_{nm}(\boldsymbol{x}')}{2B\,(m+n+1)+\mu^2}\,. \tag{4.65}$$

Since for real B we have

$$\mathcal{F}[f^{(B)}_{mn}](\boldsymbol{k})=\frac{\mathrm{i}^{\,m-n}}{B}f^{(B)}_{mn}(\boldsymbol{B}^{-1}\cdot\boldsymbol{k})\,, \tag{4.66}$$

which is proven in appendix H, and

$$\begin{aligned}\mathcal{F}[\mathsf{P}_i^2 f^{(B)}_{mn}](\boldsymbol{k})&=4B\left(m+\frac{1}{2}\right)\mathcal{F}[f^{(B)}_{mn}](\boldsymbol{k})\,,\\ \mathcal{F}[\tilde{\mathsf{P}}_i^2 f^{(B)}_{mn}](\boldsymbol{k})&=4B\left(n+\frac{1}{2}\right)\mathcal{F}[f^{(B)}_{mn}](\boldsymbol{k})\,,\end{aligned} \tag{4.67}$$

we find that Fourier transformation relates the propagator in position space to the momentum space propagator

$$\hat{\Delta}(\boldsymbol{k},\boldsymbol{k}')=\frac{1}{B^2}\Delta(\tilde{\boldsymbol{k}},\tilde{\boldsymbol{k}}')\,. \tag{4.68}$$

This is just the reflection of the classical LS-covariance proven in section 3.3 and coincides with the general expression (4.64) for $g=0$. Following [LS02a], an appropriate regularization scheme which cuts off simultaneously short distances and low momenta in a duality invariant way is to modify the propagator with the help of the operator $\mathsf{P}_i^2+\tilde{\mathsf{P}}_i^2=-\partial_i^2+\Omega^2\tilde{x}_i^2$. Let $\Lambda\in\mathbb{R}_+$ be a cut-off parameter and L a smooth cut-off function which is monotonically decreasing with $L(y)=1$ for $y<1$ and $L(y)=0$ for $y>2$. The modified propagator in position space is thus given by

$$\Delta_\Lambda(\boldsymbol{x},\boldsymbol{x}')=\langle\boldsymbol{x}|\frac{1}{\frac{1}{2}\mathsf{P}_i^2+\frac{1}{2}\tilde{\mathsf{P}}_i^2+\mu^2}L\left(\Lambda^{-2}|\mathsf{P}_i^2+\tilde{\mathsf{P}}_i^2|\right)|\boldsymbol{x}'\rangle \tag{4.69}$$

Since $\mathsf{P}_i^2+\tilde{\mathsf{P}}_i^2$ is LS-duality covariant, this is a covariant regularization of the propagator. One expects $-\partial_i^2$ to cut off high momenta and $\tilde{x}_i^2$ to regulate possible infrared divergences. This gets substantiated with help of the matrix representation.

Contrary to the previous section we adjust the matrix functions such as to diagonalize the propagator

$$\begin{aligned}\Delta_\Lambda(mn;k\ell)\quad&=\quad\frac{\delta_{m\ell}\delta_{nk}}{2B\,(m+n+1)+\mu^2}L\left(\Lambda^{-2}\left[4B(m+n+1)\right]\right)\\ &=:\quad\delta_{m\ell}\delta_{nk}\Delta_\Lambda(m,n)\,.\end{aligned} \tag{4.70}$$

The interaction vertex in matrix representation is now given by

$$v(m_1,n_1,\dots,m_4,n_4):=\int_x\left(f^{(B)}_{m_1n_1}\star_\theta f^{(B)}_{m_2n_2}\star_\theta f^{(B)}_{m_3n_3}\star_\theta f^{(B)}_{m_4n_4}\right)(\boldsymbol{x})\,. \tag{4.71}$$

The Landau functions are elements of a subspace of $\mathcal{S}(\mathbb{R}^2)$, the so called Gel'fand-Shilov space $\mathcal{S}^\alpha_\alpha(\mathbb{R}^2)$ with $\alpha \geq 1/2$ (see e.g. appendix C.1), which is closed under the star-product. We thus follow that the interaction vertex v is *well-defined.* All Feynman diagrams are now of the form

$$\sum_{m_1 n_1,\dots,m_K n_K} \prod_{i=1}^{K} \Delta_\Lambda(m_i, n_i)\,(\cdots) \tag{4.72}$$

where $(\cdots)$ is a product of interaction vertex v, depending on the regularized propagator Δ_Λ and external vertices $m_1, n_1, \dots, m_M, n_M$. Since the Δ_Λ is only nonzero for $4B(m+n+1) < 2\Lambda^2$, all Feynman diagrams are represented by finite sums, and thus constitute well-defined and LS-duality covariant Green functions $\mathcal{G}_M(m_1, n_1, \dots, m_M, n_M)$ in matrix basis.

By multiplying these expression with $f^{(B)}_{m_i n_i}(\boldsymbol{x}_i)$ for $i = 1, \dots, M$ and summing over all m_i, n_i we get back the position space Green functions. They are also well-defined and LS-duality covariant, since they are build by finite sums of well-defined covariant objects. This ends the proof of the LS-duality of the GW model at quantum level. The proof for the LSZ model is identical.

4.6 Generalization to Higher Dimensions

The $D = 2n$-dimensional LS-covariant theories are linear combinations of the operators $(\mathsf{P}^2_i)_k$ and $(\tilde{\mathsf{P}}^2_i)_k$ for $k = 1, \dots, n$, see (3.11). Since all of them commute with each other, the generalization to higher dimensions is remarkable simple. Taking $D/2 = n$ copies of the Landau functions $f^{(B_k)}_{m_k n_k}(\boldsymbol{x}_k)$ with $\boldsymbol{x}_k = (x^{2k-1}, x^{2k}) \in \mathbb{R}^2$, the products

$$f^{(\boldsymbol{B})}_{\boldsymbol{mn}}(\boldsymbol{x}) := \prod_{k=1}^{n} f^{(B_k)}_{m_k n_k}(\boldsymbol{x}_k)\,, \tag{4.73}$$

for all multi-dimensional indices $\boldsymbol{m} = (m_k), \boldsymbol{n} = (n_k) \in \mathbb{N}^n$, $\boldsymbol{B} = (B_k) \in \mathbb{R}^n_+$ and $\boldsymbol{x} = (x^i) \in \mathbb{R}^D$, obviously form an orthonormal basis for $L^2(\mathbb{R}^D)$ and are eigenfunctions of K^2_i and $\tilde{\mathsf{K}}^2_i$. The deformation matrix Θ is assumed to be in its canonical form

$$(\Theta^{ij}) = \begin{pmatrix} 0 & \theta_1 & & & \\ -\theta_1 & 0 & & 0 & \\ & & \ddots & & \\ & & & 0 & \theta_n \\ & 0 & & -\theta_n & 0 \end{pmatrix} \tag{4.74}$$

with $\theta_i \in \mathbb{R}$. The star product of two such multi-dimensional Landau functions with respect to (4.74) decouples into products of Landau functions depending on (x_{2k-1}, x_{2k}) for $k = 1, \dots, n$. If in addition $B_k = 2/\Theta_k$ for all k, then

$$\left(f^{(\boldsymbol{B})}_{\boldsymbol{mn}} \star_\Theta f^{(\boldsymbol{B})}_{\boldsymbol{m'n'}}\right)(\boldsymbol{x}) = \delta_{\boldsymbol{nm'}}\, f^{(\boldsymbol{B})}_{\boldsymbol{mn'}}(\boldsymbol{x}) \tag{4.75}$$

with $\delta_{\boldsymbol{m'n}} = \prod_{k=1}^n \delta_{m'_k n_k}$.

The generalization of the matrix model representation is straightforward. The scalar fields living on $\mathbb{R}^D$ expanded in Landau basis read

$$\begin{aligned} \phi(\boldsymbol{x}) &= \sum_{\boldsymbol{m},\boldsymbol{n}\in\mathbb{N}^n}^{\infty} f^{(2\boldsymbol{\theta}^{-1})}_{\boldsymbol{mn}}(\boldsymbol{x})\,\phi_{\boldsymbol{mn}} \\ \phi(\boldsymbol{x})^* &= \sum_{\boldsymbol{m},\boldsymbol{n}\in\mathbb{N}^n}^{\infty} f^{(2\boldsymbol{\theta}^{-1})}_{\boldsymbol{mn}}(\boldsymbol{x})\,\overline{\phi}_{\boldsymbol{mn}} \end{aligned} \tag{4.76}$$

where the coefficients are given by

$$\begin{aligned} \phi_{\boldsymbol{mn}} &= \langle f^{(2\boldsymbol{\theta}^{-1})}_{\boldsymbol{mn}} | \phi\rangle = \int \mathrm{d}^D\boldsymbol{x}\, f^{(2\boldsymbol{\theta}^{-1})}_{\boldsymbol{nm}}(\boldsymbol{x})\,\phi(\boldsymbol{x}) \\ \overline{\phi}_{\boldsymbol{mn}} &= \langle f^{(2\boldsymbol{\theta}^{-1})}_{\boldsymbol{mn}} | \overline{\phi}\rangle = \int \mathrm{d}^D\boldsymbol{x}\, f^{(2\boldsymbol{\theta}^{-1})}_{\boldsymbol{nm}}(\boldsymbol{x})\,\phi(\boldsymbol{x})^*\,. \end{aligned} \tag{4.77}$$

and $f^{(2\boldsymbol{\theta}^{-1})}_{\boldsymbol{nm}}(\boldsymbol{x})$ given by (4.73) with $B_k = 2\theta_k^{-1}$. The matrix representation of the D-dimensional LSZ model away from the self-dual point is given by

$$\begin{aligned}\mathcal{S}_{\mathsf{LSZ}} \quad &= \sum_{\boldsymbol{m},\boldsymbol{n},\boldsymbol{k},\boldsymbol{\ell}\,\in\,\mathbb{N}^n} \overline{\phi}_{\boldsymbol{mn}} G_{\boldsymbol{mn};\boldsymbol{k\ell}}\, \phi_{\boldsymbol{\ell k}} \\ &\qquad + \frac{g}{2\pi\theta} \sum_{\boldsymbol{m},\boldsymbol{n},\boldsymbol{k},\boldsymbol{\ell}\,\in\,\mathbb{N}^n} \big(\alpha\, \overline{\phi}_{\boldsymbol{mn}} \phi_{\boldsymbol{nk}}\, \overline{\phi}_{\boldsymbol{k\ell}} \phi_{\boldsymbol{\ell m}} + \beta\, \overline{\phi}_{\boldsymbol{mn}} \overline{\phi}_{\boldsymbol{nk}} \phi_{\boldsymbol{k\ell}}\, \phi_{\boldsymbol{\ell m}}\big) \ . \end{aligned} \tag{4.78}$$

with D-dimensional wave operator

$$G_{\boldsymbol{mn};\boldsymbol{k\ell}} := \sum_{i=1}^{n} \mathcal{G}_{m_i n_i;k_i \ell_i} + \mu^2 \delta_{\boldsymbol{m\ell}} \delta_{\boldsymbol{nk}} \tag{4.79}$$

and each $\mathcal{G}_{m_i n_i;k_i \ell_i}$ given by the *massless*, two-dimensional wave operator (4.45). Any result of this chapter can now formally be generalized to higher dimensions by substituting multi-indices $\boldsymbol{m}, \boldsymbol{n}, \ldots \in \mathbb{N}^n$ for usual one-dimensional indices $m, n, \ldots \in \mathbb{N}$.

5 LS-Covariant NCQFTs in Minkowski Spacetime

The introduction of an external background field proved very useful in Euclidean NCQFT, making the theory covariant under LS-duality. Furthermore the occurring wave operators K_i^2 and $\tilde{\mathsf{K}}_i^2$ have discrete spectra such that the corresponding models can be handled properly with help of the matrix basis. However, in passing from the Euclidean LS-covariant NCQFTs to Minkowski signature, the main feature of the wave operators changes dramatically. The same background field, being magnetic in Euclidean metric, now plays the role as an electric field.

The presence of an electric-like external field implies a qualitative change compared to the magnetic field case, due to the work which is done on the particles by the field. The electric field accelerates and splits virtual dipole pairs leading to pair production. This is reflected in the spectra of the Hamiltonian and of the wave operators, being now the whole real axis and unbounded from below. The main problem for us is, that the discrete spectrum of the wave operator was essential for the model to have a matrix representation in form of a countable infinite set of eigenfunctions which solve the projector property. Surely the Landau basis can again be used to map this model onto a matrix model, however the free part will not be diagonalized, which was part of the proof of the LS-covariance at quantum level. A different matrix basis has to be found, which is tailored for the Minkowskian version of the LS-covariant noncommutative field theories.

In the following we will introduce the LS-covariant models in Minkowski spacetime. We derive their relation to the inverted harmonic oscillator which possesses the conjectured continuous spectrum. By computing its eigenfunctions a possible matrix expansion is identified with a *resonance expansion*. Finally we define the quantum field theories for both approaches, the continuous and the discrete one. To make the latter well-defined a special regularization will be introduced.

5.1 LS-Covariant Models in Minkowski Spacetime

We will again work in $D = 2n$ dimensions. Vectors will now be indicated by Greek indices $\mu, \nu, \ldots$ ranging from 0 to $d = D - 1$. The signature is given by $(1, -1, \ldots, -1)$.

The general LSZ-model in D dimensional spacetime is given by the action $\mathcal{S}_0 + \mathcal{S}_{\mathsf{int}}$ with

$$\mathcal{S}_0 = \int \mathrm{d}^D\boldsymbol{x}\, \phi^*(\boldsymbol{x}) \left(\sigma \mathsf{K}_\mu^2 + (1-\sigma)\tilde{\mathsf{K}}_\mu^2 - \mu^2\right) \phi(\boldsymbol{x}) \tag{5.1}$$

$$\mathcal{S}_{\mathsf{int}} = -g \int \mathrm{d}^D\boldsymbol{x}\, \left[\alpha\, (\phi^* \star \phi \star \phi^* \star \phi)(\boldsymbol{x}) + \beta\, (\phi^* \star \phi^* \star \phi \star \phi)(\boldsymbol{x})\right] \tag{5.2}$$

with generalized momenta

$$\begin{aligned} \mathsf{K}_\mu &= \mathrm{i}\,\partial_\mu - F_{\mu\nu}x^\nu \\ \tilde{\mathsf{K}}_\mu &= \mathrm{i}\,\partial_\mu + F_{\mu\nu}x^\nu\,, \end{aligned} \tag{5.3}$$

obeying the commutation relations

$$[\mathsf{K}_\mu, \mathsf{K}_\nu] = 2\,\mathrm{i}\,F_{\mu\nu} \quad , \quad [\tilde{\mathsf{K}}_\mu, \tilde{\mathsf{K}}_\nu] = -2\,\mathrm{i}\,F_{\mu\nu}\,. \tag{5.4}$$

and $[\mathsf{K}_\mu, \tilde{\mathsf{K}}_\nu] = 0$. The coordinate system will be chosen such that the $D \times D$ dimensional deformation matrix Θ takes the canonical skew-symmetric form

$$(\Theta^{\mu\nu}) = \begin{pmatrix} 0 & \theta_1 & & & & \\ -\theta_1 & 0 & & & 0 & \\ & & \ddots & & & \\ & & & & 0 & \theta_n \\ & 0 & & & -\theta_n & 0 \end{pmatrix} \tag{5.5}$$

with $\theta_k > 0$ for all k. The electromagnetic tensor $F_{\mu\nu}$ is given by

$$(F_{\mu\nu}) = \begin{pmatrix} 0 & E & & & & & & \\ -E & 0 & & & & & 0 & \\ & & 0 & B_2 & & & & \\ & & -B_2 & 0 & & & & \\ & & & & \ddots & & & \\ & & & & & & 0 & B_n \\ & 0 & & & & & -B_n & 0 \end{pmatrix} \tag{5.6}$$

for $E, B_k > 0$ and $E\theta_1/2 = B_k\theta_k/2 = \Omega$ for all k and $0 < \Omega \leq 1$. The $2n$-dimensional wave operators again break up into n parts

$$\begin{aligned} \mathsf{K}_\mu^2 &= \sum_{k=1}^{n} (\mathsf{P}_\mu^2)_k \\ \tilde{\mathsf{K}}_\mu^2 &= \sum_{k=1}^{n} (\tilde{\mathsf{P}}_\mu^2)_k \end{aligned} . \tag{5.7}$$

The negative of the operators

$$\begin{aligned} (\mathsf{P}_\mu^2)_k &= (\partial_{2k-2}^2 + \partial_{2k-1}^2) + 2\,\mathrm{i}\, B_k (x^{2k-1}\partial_{2k-2} - x^{2k-2}\partial_{2k-1}) - B_k^2 (x_{2k-2}^2 + x_{2k-1}^2) \\ (\tilde{\mathsf{P}}_\mu^2)_k &= (\partial_{2k-2}^2 + \partial_{2k-1}^2) - 2\,\mathrm{i}\, B_k (x^{2k-1}\partial_{2k-2} - x^{2k-2}\partial_{2k-1}) - B_k^2 (x_{2k-2}^2 + x_{2k-1}^2) \end{aligned} \tag{5.8}$$

for $k = 2, \ldots, n$ describe two dimensional Euclidean Klein-Gordon fields moving in an external magnetic background, already investigated in chapter 4, while

$$\begin{aligned} (\mathsf{P}_\mu^2)_1 &= -(\partial_0^2 - \partial_1^2) - 2\,\mathrm{i}\, E(x^1\partial_0 + x^0\partial_1) - E^2(x_0^2 - x_1^2) \\ (\tilde{\mathsf{P}}_\mu^2)_1 &= -(\partial_0^2 - \partial_1^2) + 2\,\mathrm{i}\, E(x^1\partial_0 + x^0\partial_1) - E^2(x_0^2 - x_1^2) \end{aligned} \tag{5.9}$$

describe 1+1 dimensional KG fields moving in a constant *electric* background with field strengths $\pm 2E$, respectively. Again, all two-dimensional operators $(\mathsf{P}_\mu^2)_k$ and $(\tilde{\mathsf{P}}_\mu^2)_k$ mutually commute such that diagonalization of the full wave operators amounts to diagonalizing each of its 2 dimensional parts independently. Eigenfunctions of the operators (5.8) are the Landau functions. What is missing are the eigenfunctions of (5.9).

The first important observation is that the operators (5.9) can not be obtained from its Euclidean counterpart by an *ordinary* Wick rotation $t \to \mathrm{i}\, t$. Some additional signs have changed, showing that for our theories the rotation of time has to be accompanied by the transformations $B \to -\mathrm{i}\, E$. This is not surprising, since this model can be viewed as field theory on a curved, non-static spacetime, for which this is a generic feature [DeW75]. Another characteristic of those theories is that the degeneracy of the different equivalent definitions of the Feynman propagator is resolved [CR77]. We will come back to this problem in section 7.1. The eigenfunctions of (5.9) will be determined in section 5.3.

The extra transformation of the magnetic field strength is in concordance with the fact, that in order to ensure the relation

$$[x^0, x^i] = \mathrm{i}\,\Theta^{0i} \tag{5.10}$$

for Euclidean and Minkowskian space, the deformation parameter Θ^{0i} has to transform accordingly to compensate the phase coming from the Wick rotation. For LS-invariant theories, the deformation matrix is related to the field strength, which in turn implies a rotation of the field strength.

The Minkowskian Grosse-Wulkenhaar model in D spacetime dimensions is again the general LSZ model for $\sigma = 1/2$ involving real fields. Exactly as in the LSZ case, the D dimensional wave operator reduces to a sum of $n-1$ Euclidean GW wave operators plus a two dimensional GW wave operator in Minkowski signature

$$\frac{1}{2}(\mathsf{P}_\mu^2)_1 + \frac{1}{2}(\tilde{\mathsf{P}}_\mu^2)_1 - \mu^2 = -(\partial_0^2 - \partial_1^2) - \Omega^2(x_0^2 - x_1^2) - \mu^2 \tag{5.11}$$

with frequency $\Omega = E\theta/2$. The main difference, beside the Minkowskian signature, is an extra minus sign in front of the Ω-term. The corresponding wave operator is an harmonic oscillator with imaginary frequency, known as *inverted harmonic oscillator*. We will come across this oscillator in the next section.

5.2 Mapping onto the Inverted Harmonic Oscillator

We need to find the eigenfunctions of the part of the action depending on the coordinates $\boldsymbol{x}_1 := (x^0, x^1) = (t, x)$, given by the wave operator $\sigma(\mathsf{P}_\mu^2)_1 + (1-\sigma)(\tilde{\mathsf{P}}_\mu^2)_1 - \mu^2$. In the following, we work in 1+1 dimensions and drop the index "1" at the wave operators, the coordinates and the deformation parameter. As in section 4.1, we use the Weyl-Wigner correspondence to solve the simultaneous eigenvalue equations

$$\begin{aligned} \mathsf{P}_\mu^2\, f_{mn}(\boldsymbol{x}) &= \lambda_{mn} f_{mn}(\boldsymbol{x})\,, \\ \tilde{\mathsf{P}}_\mu^2\, f_{mn}(\boldsymbol{x}) &= \tilde{\lambda}_{mn} f_{mn}(\boldsymbol{x})\,. \end{aligned} \tag{5.12}$$

Noting that

$$\begin{aligned} E^2x^2 \star_\theta f(\boldsymbol{x}) = &\left(E^2x^2 - \mathrm{i}\, E^2\theta x\partial_t - \frac{1}{4}E^2\theta^2\partial_t^2\right) f(\boldsymbol{x})\,, \\ E^2t^2 \star_\theta f(\boldsymbol{x}) = &\left(E^2t^2 + \mathrm{i}\, E^2\theta t\partial_x - \frac{1}{4}E^2\theta^2\partial_x^2\right) f(\boldsymbol{x})\,, \end{aligned} \tag{5.13}$$

we find that in the Minkowski case the action of P_μ^2 and $\tilde{\mathsf{P}}_\mu^2$ can be represented as a left- and right-star action of the classical function $E^2(x^2-t^2)$:

$$\begin{aligned} E^2(x^2-t^2) \star_\theta g(\boldsymbol{x}) \quad &= \quad \left[E^2(x^2-t^2) - \mathrm{i}\, E^2\theta(x\partial_t + t\partial_x) - \frac{1}{4}E^2\theta^2(\partial_t^2 - \partial_x^2)\right] g(\boldsymbol{x}) \\ &\overset{\theta=2/E}{=} \quad \mathsf{P}_\mu^2\, g(\boldsymbol{x})\,. \end{aligned} \tag{5.14}$$

Analogously one finds $\tilde{\mathsf{P}}_\mu^2\, g(\boldsymbol{x}) = g(\boldsymbol{x}) \star_{2/E} E^2(x^2-t^2)$. To compare to the Euclidean version we have to identify $B \equiv E$ and the ordered coordinate pairs $(x,y)_{Eucl} \equiv (t,x)_{Mink}$ and find

$$E^2(x^2 \pm t^2) \star_{2/E} f(\boldsymbol{x}) = \begin{cases} \mathsf{P}_i^2\, f(\boldsymbol{x}) \\ \mathsf{P}_\mu^2\, f(\boldsymbol{x}) \end{cases}. \tag{5.15}$$

Defining the Weyl symbols

$$\mathsf{W}\left[\sqrt{2}Et\right] = \hat{\mathbf{q}} \quad , \quad \mathsf{W}\left[\sqrt{2}Ex\right] = \hat{\mathbf{p}} \quad \text{and} \quad \mathsf{W}\left[f_{mn}\right] = \hat{\mathbf{f}}_{mn}\,, \tag{5.16}$$

we find the Heisenberg algebra

$$[\hat{\mathbf{q}}, \hat{\mathbf{p}}] = 2E^2[t,x]_{\star_{2/E}} = \mathrm{i}\, 4E\,, \tag{5.17}$$

and the eigenvalue equation (5.12) can be expressed as

$$\begin{aligned} \mathsf{P}_\mu^2\, f_{mn}(\boldsymbol{x}) = \mathsf{W}\left[\hat{\mathbf{H}}_i\hat{\mathbf{f}}_{mn}\right] = \lambda_{mn} f_{mn}(\boldsymbol{x})\,, \\ \tilde{\mathsf{P}}_\mu^2\, f_{mn}(\boldsymbol{x}) = \mathsf{W}\left[\hat{\mathbf{f}}_{mn}\hat{\mathbf{H}}_i\right] = \tilde{\lambda}_{mn} f_{mn}(\boldsymbol{x})\,. \end{aligned} \tag{5.18}$$

The operator $\hat{\mathbf{H}}_i = \frac{1}{2}(\hat{\mathbf{p}}^2 - \hat{\mathbf{q}}^2)$ is known as *inverted harmonic oscillator*. In $\hat{\mathbf{q}}$ eigenbasis it has the form

$$\hat{\mathbf{H}}_i \sim \frac{1}{2}(-\partial_q^2 - \gamma q^2) \tag{5.19}$$

thus it is a harmonic oscillator with imaginary frequency $\pm\mathrm{i}\,\gamma$. Due to the minus sign in front of the potential q^2, it describes a one dimensional quantum mechanical particle in a potential which is unbounded from below! This is an unbounded operator in $L^2(\mathbb{R})$ and has a continuous spectrum extending over the whole real axis $\sigma(\hat{\mathbf{H}}_i) = \mathbb{R}$ as already anticipated above. This shows that the necessary ingredients which led to the matrix basis in Euclidean space are not given. In the following section we will investigate the continuous eigenfunctions and demonstrate, how to squeeze out the matrix nature of the model.

5.3 Eigenfunction Expansion and Resonances

In order to figure out which possibilities we have to describe the LS-covariant models, we will now find the eigenfunctions of the inverted harmonic oscillator. An analytical continuation of the eigenvectors to the complex energy plane will reveal a discrete pole structure, which allows us to construct a matrix basis expansion in terms of *resonances*. These two competing approaches, based on the continuous an the discrete resonance expansion, are analyzed and compared afterwards.

The inverted harmonic oscillator is parity invariant, thus each eigenvalue is two-fold degenerated as indicated through an additional index $\pm$ carried by the eigenfunctions. The eigenvalue equation

$$\frac{1}{2}(-\partial_q^2 - \gamma q^2)\chi_\pm^{\mathcal{E}}(q) = \mathcal{E}\chi_\pm^{\mathcal{E}}(q) \tag{5.20}$$

with $\mathcal{E} \in \mathbb{R}$ gets rearranged by substituting $z = \sqrt{2\,\mathrm{i}\,\gamma}q$

$$\left(\partial_z^2 + \nu + \frac{1}{2} - \frac{z^2}{4}\right)\chi_\pm^{\mathcal{E}}(z) = 0\,, \tag{5.21}$$

where

$$\nu = -\,\mathrm{i}\,\frac{\mathcal{E}}{\gamma} - \frac{1}{2}\,. \tag{5.22}$$

The differential equation (5.21) is solved by parabolic cylinder functions $D_\nu(z)$ which are defined by

$$D_\nu(z) := \frac{1}{\Gamma(-\nu)}\,\mathrm{e}^{-\frac{1}{4}z^2}\int_0^\infty \mathrm{d}t\,\mathrm{e}^{-zt}\,\mathrm{e}^{-\frac{1}{2}t^2}t^{-\nu-1}\,. \tag{5.23}$$

In particular, every solution is a linear combination of the functions $D_\nu(z)$, $D_\nu(-z)$, $D_{-\nu-1}(\mathrm{i}\,z)$ and $D_{-\nu-1}(-\,\mathrm{i}\,z)$. Only two of them are linearly independent. One such complete set of eigenfunctions are given by [Chr04]

$$\chi_\pm^{\mathcal{E}}(q) = \frac{C}{\sqrt{2\pi\gamma}}\,\mathrm{i}^{\frac{\nu}{2}+\frac{1}{4}}\Gamma(\nu+1)D_{-\nu-1}\left(\mp\sqrt{-2\,\mathrm{i}\,\gamma}q\right)\,, \tag{5.24}$$

where $C = (\gamma/2\pi^2)^{1/4}$. Taking the other two parabolic cylinder functions, we get the complex conjugated of the $\chi_s^{\mathcal{E}}$. These functions satisfy the orthonormality and completeness relations [Chr04]

$$\int_{\mathbb{R}} \mathrm{d}q\,\chi_s^{\mathcal{E}_1}(q)^\star\chi_{s'}^{\mathcal{E}_2}(q) = \delta_{ss'}\delta(\mathcal{E}_1 - \mathcal{E}_2) \quad \text{and} \quad \sum_{s=\pm}\int_{\mathbb{R}} \mathrm{d}\mathcal{E}\,\chi_s^{\mathcal{E}}(q)^\star\chi_s^{\mathcal{E}}(q') = \delta(q-q') \tag{5.25}$$

and belong to the space of tempered distributions $\mathcal{S}'(\mathbb{R})$. The *Gel'fand-Maurin theorem* now ensures that the operator $\hat{\mathbf{H}}_i$ can be decomposed on $\mathcal{S}(\mathbb{R})$ into these eigenfunctions.[1] This means each field in $\psi \in \mathcal{S}(\mathbb{R})$ is given by

$$|\psi\rangle = \sum_{s=\pm}\int_{\mathbb{R}} \mathrm{d}\mathcal{E}\,\psi_s^{\mathcal{E}}\,|\chi_s^{\mathcal{E}}\rangle \quad \text{with} \quad \psi_s^{\mathcal{E}} = \int_{\mathbb{R}} \mathrm{d}q\,\psi(q)\,\chi_s^{\mathcal{E}}(q)^*\,, \tag{5.26}$$

and $\hat{\mathbf{H}}_i$ has the spectral decomposition

$$\hat{\mathbf{H}}_i = \sum_{s=\pm}\int_{\mathbb{R}} \mathrm{d}\mathcal{E}\,\mathcal{E}\,|\chi_s^{\mathcal{E}}\rangle\langle\chi_s^{\mathcal{E}}|\,. \tag{5.27}$$

[1] The eigenfunctions $\chi_s^{\mathcal{E}}$ are not in the Hilbert space $L^2(\mathbb{R})$, which is a characteristic feature of unbounded operators. The mathematical framework for dealing with unbounded operators is given by the Gel'fand-Maurin theorem. Let $\hat{\mathbf{A}}$ be an unbounded self-adjoint operator defined on an infinite-dimensional Hilbert space $\mathcal{H}$. Roughly it says that if a *rigged Hilbert space* can be found, that is a triplet of spaces $\Phi \subset \mathcal{H} \subset \Phi'$, where Φ is a dense, topological vector subspace of $\mathcal{H}$ and Φ' its topological dual, then, having for each value from the spectrum of $\hat{\mathbf{A}}$ an eigenvector $F \in \Phi'$, we can expand $\hat{\mathbf{A}}$ (restricted to Φ) and each $\phi \in \Phi$ in this eigenbasis. This theorem is the mathematical basis for quantum mechanics. See [dlM05] for an introduction.

with abuse of Dirac's bra-ket notation.

Now the eigenfunctions $\chi_s^{\mathcal{E}}$ possess a peculiar analytical structure, if the energy $\mathcal{E}$ gets analytically continued to the complex plane. It has poles on the negative imaginary axis, and furthermore, its residues at these poles are harmonic oscillator eigenfunctions corresponding to imaginary "eigenvalues"! To see this let's state the following lemma proved in [Chr04]

Lemma 5.1. *The parabolic cylinder functions $D_\lambda(z)$ are analytic functions of $\lambda \in \mathbb{C}$.*

The analytical structure of the functions (5.24) is thus entirely governed by the gamma-functions. Since the only singularities of $\Gamma(\lambda)$ are simple poles at $\lambda = -n, n \in \mathbb{N}_0$ with residues

$$\mathrm{Res}_{\lambda=-n}\left(\Gamma(\lambda)\right) = \frac{(-1)^n}{n!} \tag{5.28}$$

and $\mathcal{E} = \mathrm{i}\,\gamma\left(\nu + \frac{1}{2}\right)$, we see that $\chi_\pm^{\mathcal{E}}$ has poles at $\mathcal{E} = -\mathrm{i}\,\gamma\left(n + \frac{1}{2}\right)$ with residues

$$\mathrm{Res}_{\mathcal{E}=-\mathrm{i}\,\gamma\left(n+\frac{1}{2}\right)}\left(\chi_\pm^{\mathcal{E}}(q)\right) \sim \frac{(-1)^n}{n!}\,\mathrm{i}^{-\frac{n}{2}-\frac{1}{4}} D_n\left(\mp\sqrt{-2\,\mathrm{i}\,\gamma} q\right). \tag{5.29}$$

Now using

$$D_n(z) = 2^{-n/2}\,\mathrm{e}^{-z^2/4} H_n(z/\sqrt{2}) \tag{5.30}$$

for $n \in \mathbb{N}_0$, we find

$$\mathrm{Res}_{\mathcal{E}=-\mathrm{i}\,\gamma\left(n+\frac{1}{2}\right)}\left(\chi_\pm^{\mathcal{E}}(q)\right) \sim (\mp 1)^n f_n^-(q) \tag{5.31}$$

with

$$f_n^-(q) = \left(\frac{\sqrt{-\mathrm{i}\,\gamma}}{2^n n! \sqrt{\pi}}\right)^{1/2} \mathrm{e}^{\,\mathrm{i}\,\frac{\gamma}{2}q^2} H_n(\sqrt{-\mathrm{i}\,\gamma} q)\,. \tag{5.32}$$

One should note that starting with the complex conjugated functions $(\chi_\pm^{\mathcal{E}})^*$ we would have found poles in the upper complex half plane, with residues proportional to $f_n^-|_{\gamma\to-\gamma} =: f_n^+$. The interpretation of the different sets of functions will become clear shortly.

One might be reminded of the eigenfunctions of the ordinary harmonic oscillator, which are given by

$$\phi_n(q) = \left(\frac{\sqrt{\gamma}}{2^n n! \sqrt{\pi}}\right)^{1/2} \mathrm{e}^{-\frac{\gamma}{2}q^2} H_n(\sqrt{\gamma} q)\,, \tag{5.33}$$

and show up in the Euclidean case. As said before, the inverted harmonic oscillator emerges by inserting an imaginary frequency $\pm\,\mathrm{i}\,\gamma$ into the usual harmonic oscillator, which also transforms the harmonic oscillator functions (5.33) into $f_n^\pm$. Though these are not eigenfunctions in the usual sense, they appear as residues of the proper eigenfunctions $\chi_\pm^{\mathcal{E}}$. One can easily verify that the $f_n^\pm$ are not ordinary eigenfunctions of $\hat{\mathbf{H}}_i$ by looking at the "eigenvalue equation"

$$\hat{\mathbf{H}}_i f_n^\pm(q) = \pm\,\mathrm{i}\,\gamma\left(n + \frac{1}{2}\right) f_n^\pm(q)\,, \tag{5.34}$$

which follows directly from the defining equation for Hermite polynomials. This equation seems to contradict the well-known fact that Hermitian operators have real eigenvalues. But the $f_n^\pm$ are in $\mathcal{S}'(\mathbb{R})$ and $\hat{\mathbf{H}}_i$ is not Hermitian on these states:

$$\langle f_n^\pm | \hat{\mathbf{H}}_i f_n^\pm \rangle \neq \langle \hat{\mathbf{H}}_i f_n^\pm | f_n^\pm \rangle \tag{5.35}$$

due to non-vanishing boundary terms. Apart from this, the $f_n^\pm$ are not normalizable in L^2-norm:

$$\begin{aligned} \langle f_n^\pm | f_n^\pm \rangle &= \int_{\mathbb{R}} \mathrm{d}q\, f_n^\mp(q)\, f_n^\pm(q) \\ &\sim \int_{\mathbb{R}} \mathrm{d}q\, H_n(\sqrt{\mp\,\mathrm{i}\,\gamma}\, q)\, H_n(\sqrt{\pm\,\mathrm{i}\,\gamma}\, q) = \infty\,. \end{aligned} \tag{5.36}$$

Such states are known as *resonance states* or *Gamow states*, which were first introduced to describe decay phenomena in nuclei. They correspond to complex eigenvalues of the Hamiltonian and are a characteristic feature of open quantum systems. The imaginary part of the "Hamiltonian expectation value" of the resonant state determines the momentum flux out of the system, which is proportional to $\langle f_n^\pm | f_n^\pm \rangle$. This expression is infinite, which mirrors the fact that in an infinite volume an infinite amount of real particle/anti-particle pairs are produced per unit time. Resonant states always occur as resonant/anti-resonant pairs, which in our case are the pairs of poles $\pm\,\mathrm{i}\,\gamma(n+1/2)$. For an overview see [CG04, HSNP08].

How can these functions nevertheless help us constructing a diagonal matrix expansion of our models? First of all note that the L^2 scalar product of f_n^+ with f_n^- can be defined as

$$\langle f_n^+ | f_m^- \rangle = \delta_{nm}\,, \tag{5.37}$$

by an analytical continuation of the identity $\langle \phi_n | \phi_m \rangle = \delta_{nm}$ to imaginary frequencies. Thus they form a mathematical structure called *bi-orthogonal system*. The naive answer is then that by closing the integration contour of (5.26) and (5.27) in the lower complex half plane we pick up the poles with help of the residue theorem. This technique is well known in the physics literature, called *resonance expansion*. Using

$$(\mp 1)^n D_{-n-1}(-\sqrt{2\,\mathrm{i}\,\gamma}q) + D_{-n-1}(\sqrt{2\,\mathrm{i}\,\gamma}q) = \frac{\sqrt{2\pi}}{n!}(-\,\mathrm{i}\,)^n 2^{-n/2}\,\mathrm{e}^{\,\mathrm{i}\,\frac{\gamma}{2}q^2} H_n(\sqrt{-\,\mathrm{i}\,\gamma}q) \tag{5.38}$$

we find

$$-\,2\pi\,\mathrm{i} \sum_{s=\pm} \mathrm{Res}_{\mathcal{E}=-E_n}\left(\langle q|\chi_s^{\mathcal{E}}\rangle\langle\chi_s^{\mathcal{E}}|q'\rangle\right) = f_n^-(q) f_n^-(q') \tag{5.39}$$

and thus get the following formal expansions

$$\begin{aligned} |\psi\rangle &= \sum_{n=0}^{\infty} |f_n^-\rangle\langle f_n^+|\psi\rangle \\ \hat{\mathsf{H}}_i &= \sum_{n=0}^{\infty} (-\,\mathrm{i}\,)\gamma(n+1/2)|f_n^-\rangle\langle f_n^+|\,. \end{aligned} \tag{5.40}$$

Note that in both expansions different functions appear in the kets and bras. This is in concordance with the pairing defined in (5.37), which is only well-defined for f^--kets with f^+-bras or vice versa.

The resonance expansion should be allowed for those functions, for which the integrand of the eigenvector expansion vanishes faster than $1/r$ in the lower complex half plane, if r determines the distance to the origin. Secondly, since there are infinitely many poles scattered over a non-compact region, we have to make sure that the arising sum converges. The ordinary Landau basis were naturally defined on the Schwartz space, which as its most prominent representative has the Gaussian $\psi(q) = \mathrm{e}^{-bq^2}$. Since this Gaussian already features all the problems we will encounter, we will try to expand it in the f_n^- basis. Instead of verifying that the integrand vanishes faster than $1/r$ in the complex plane (which is possible), we will expand this function directly in f_n^--basis and check whether

$$\psi(q) \overset{?!}{=} \sum_{n=0}^{\infty} f_n^-(q)\,\langle f_n^+|\psi\rangle\,. \tag{5.41}$$

Though we will find that the resulting series is *not absolutely convergent*, it inevitably tells us how to overcome these problems.

As is shown in proposition B.1 the f_n^- can be represented as[2]

$$f_n^- = \left(\frac{\sqrt{-\,\mathrm{i}\,\gamma}}{2^n n!\sqrt{\pi}}\right)^{1/2} (2\,\mathrm{i}\,)^{n/2} \int_{-\infty}^{\infty} \mathrm{d}a\,(-1)^n \delta^{(n)}(a)\,\mathrm{e}^{\,\mathrm{i}\,S(x,a)} \tag{5.42}$$

with $S(x,a) = \frac{\gamma}{2}x^2 - \sqrt{2\gamma}xa + \frac{a^2}{2}$ and $\gamma > 0$. The coefficients $\psi_n = \int_q f_n^-(q)\psi(q)$ are given by

$$\psi_n = \left(\frac{\sqrt{-\,\mathrm{i}\,\gamma}}{\sqrt{\pi}}\right)^{1/2} \frac{\mathrm{i}^{\,n/2}}{\sqrt{n!}} \int \mathrm{d}x \int \mathrm{d}a\,(-1)^n \delta^{(n)}(a)\,\mathrm{e}^{\,\mathrm{i}\,S(x,a)}\,\mathrm{e}^{-b\,x^2}\,. \tag{5.43}$$

[2]In the notation used in the appendix we have $f_n^- \equiv f_n^{(-\,\mathrm{i}\,\gamma)}$ with $\gamma > 0$.

The x-integration is Gaussian and can be performed

$$\int_x \mathrm{e}^{\,\mathrm{i}\,\frac{\gamma}{2}x^2-\mathrm{i}\,\sqrt{2\gamma}xa+\frac{\mathrm{i}}{2}a^2-b\,x^2} = \sqrt{\frac{\pi}{b-\mathrm{i}\,\gamma/2}}\exp\left\{\frac{\mathrm{i}}{2}a^2\left(\frac{b+\mathrm{i}\,\gamma/2}{b-\mathrm{i}\,\gamma/2}\right)\right\} \tag{5.44}$$

leading to

$$\psi_n = \left(\frac{\sqrt{-\mathrm{i}\,\gamma}}{\sqrt{\pi}}\right)^{1/2}\frac{\mathrm{i}^{\,n/2}}{\sqrt{n!}}\sqrt{\frac{\pi}{b-\mathrm{i}\,\gamma/2}}\left(\frac{\mathrm{i}}{2}\frac{b+\mathrm{i}\,\gamma/2}{b-\mathrm{i}\,\gamma/2}\right)^{n/2}\int_a \delta(a)\partial_a^n\,\mathrm{e}^{\,a^2}\,. \tag{5.45}$$

The a integration follows from

$$\int_a \delta(a)\partial_a^n\,\mathrm{e}^{\,a^2} = \partial_a^n\,\mathrm{e}^{\,a^2}\Big|_{a=0} = \begin{cases} \frac{n!}{(n/2)!} & n \text{ even} \\ 0 & n \text{ odd}\end{cases}\,, \tag{5.46}$$

hence

$$\psi_n = \left(\frac{\sqrt{-\mathrm{i}\,\gamma}}{\sqrt{\pi}}\right)^{1/2}\frac{\mathrm{i}^{\,n/2}}{\sqrt{n!}}\sqrt{\frac{\pi}{b-\mathrm{i}\,\gamma/2}}\left(\frac{\mathrm{i}}{2}\frac{b+\mathrm{i}\,\gamma/2}{b-\mathrm{i}\,\gamma/2}\right)^{n/2}\frac{n!}{(n/2)!}\,. \tag{5.47}$$

Putting this back into the expansion yields

$$\begin{aligned}
&\sum_{n=0}^{\infty}\psi_n\,f_n^-(x)\\
&= \sum_{n=0}^{\infty}\left(\frac{\sqrt{-\mathrm{i}\,\gamma}}{2^n n!\sqrt{\pi}}\right)^{1/2}\mathrm{e}^{\,\mathrm{i}\,\frac{\gamma}{2}x^2}H_n\left(\sqrt{-\mathrm{i}\,\gamma}x\right)\\
&\quad\times\left(\frac{\sqrt{-\mathrm{i}\,\gamma}}{\sqrt{\pi}}\right)^{1/2}\frac{\mathrm{i}^{\,n/2}}{\sqrt{n!}}\sqrt{\frac{\pi}{b-\mathrm{i}\,\gamma/2}}\left(\frac{\mathrm{i}}{2}\frac{b+\mathrm{i}\,\gamma/2}{b-\mathrm{i}\,\gamma/2}\right)^{n/2}\frac{n!}{(n/2)!}\begin{cases}1 & n \text{ even}\\ 0 & n \text{ odd}\end{cases}\\
&= \sum_{k=0}^{\infty}\sqrt{\frac{-\mathrm{i}\,\gamma}{b-\mathrm{i}\,\gamma/2}}\left(\frac{\mathrm{i}}{2}\right)^{2k}\left(\frac{b+\mathrm{i}\,\gamma/2}{b-\mathrm{i}\,\gamma/2}\right)^{k}\frac{1}{k!}\,\mathrm{e}^{\,\mathrm{i}\,\frac{\gamma}{2}x^2}H_{2k}\left(\sqrt{-\mathrm{i}\,\gamma}x\right)\,.
\end{aligned} \tag{5.48}$$

The sum can be performed using equation (49.4.4) from [Han75]:

$$\sum_{k=0}^{\infty}\frac{t^k}{k!}H_{2k}(z) = (1+4t)^{-1/2}\exp\left(\frac{4tz^2}{1+4t}\right)\,. \tag{5.49}$$

This formula is clearly not valid for all $t \in \mathbb{C}$. Using the asymptotic behavior of the Hermite function for $n \to \infty$ [MOS66]

$$H_n(x) \sim \frac{n!}{(n/2)!}\,\mathrm{e}^{\sqrt{2n}|\Im\mathfrak{m}x|} \tag{5.50}$$

and Stirling's formula $n! \sim n^n\,\mathrm{e}^{-n}$ we find

$$\left|\frac{t^k}{k!}H_{2k}(x)\right| \sim |t|^k\,\mathrm{e}^{\,2k\ln 2k-2k\ln k}\,\mathrm{e}^{\sqrt{4k}|\Im\mathfrak{m}x|} \sim (4|t|)^k\,\mathrm{e}^{\sqrt{4k}|\Im\mathfrak{m}x|} \tag{5.51}$$

In order to get an absolutely convergent series we have to ensure that $|t| < 1/4$. This is however not fulfilled in our case, since we have $|t| = 1/4$. This shows us that the Schwartz space is too big for our purpose.

The problem might be circumvented by considering an even smaller space, like the space of smooth functions with compact support. The expansion on $\mathcal{S}(\mathbb{R})$ might then be defined in some limiting procedure. But, since we are lying exactly at the edge of the convergence radius, a natural procedure is to make γ slightly imaginary such that $|t| < 1/4$ and we can proceed with summing up. We have

$$z = \sqrt{-\mathrm{i}\,\gamma}x \quad , \quad t = -\frac{1}{4}\frac{b+\mathrm{i}\,\gamma/2}{b-\mathrm{i}\,\gamma/2}\,, \tag{5.52}$$

which gives us the relevant combinations

$$\begin{aligned}\frac{1}{4t}+1 &= \frac{-b+\mathrm{i}\,\gamma/2+b+\mathrm{i}\,\gamma/2}{b+\mathrm{i}\,\gamma/2}=\frac{\mathrm{i}\,\gamma}{b+\mathrm{i}\,\gamma/2}\\ \frac{4tz^2}{1+4t} &= \frac{(b+\mathrm{i}\,\gamma/2)(-\mathrm{i}\,\gamma)x^2}{\mathrm{i}\,\gamma}=-\mathrm{i}\,\frac{\gamma}{2}x^2-bx^2\\ 1+4t &= \frac{-b-\mathrm{i}\,\gamma/2+b-\mathrm{i}\,\gamma/2}{b-\mathrm{i}\,\gamma/2}=\frac{-\mathrm{i}\,\gamma}{b-\mathrm{i}\,\gamma/2}\,.\end{aligned} \tag{5.53}$$

Inserting into the sum yields

$$\begin{aligned}&\sum_{n=0}^{\infty} f_n^-(x)\psi_n\\ &= \sum_{k=0}^{\infty}\sqrt{\frac{-\mathrm{i}\,\gamma}{b-\mathrm{i}\,\gamma/2}}\left(\frac{\mathrm{i}}{2}\right)^{2k}\left(\frac{b+\mathrm{i}\,\gamma/2}{b-\mathrm{i}\,\gamma/2}\right)^{k}\frac{1}{k!}\,\mathrm{e}^{\,\mathrm{i}\,\frac{\gamma}{2}x^2}H_{2k}\left(\sqrt{-\mathrm{i}\,\gamma}x\right)\\ &= \sqrt{\frac{-\mathrm{i}\,\gamma}{b-\mathrm{i}\,\gamma/2}}(1+4t)^{-1/2}\,\mathrm{e}^{\,\mathrm{i}\,\frac{\gamma}{2}x^2}\,\mathrm{e}^{-\mathrm{i}\,\frac{\gamma}{2}x^2-bx^2}\\ &= \mathrm{e}^{-b\,x^2}\,,\end{aligned} \tag{5.54}$$

giving us back the Gaussian we started with. Most importantly, this result is independent of γ, thus putting γ back on the real line is no problem at the end. The resonance expansion can thus be understood with help of the regularization $\mathrm{i}\,\gamma \to \mathrm{e}^{\,\mathrm{i}\,\vartheta}\gamma$ with $0<\vartheta<\pi/2$, while $-\pi/2<\vartheta<0$ corresponds to the analog regularized expansion in f_n^+-basis. The case $\vartheta=0$ is exactly the harmonic oscillator case (5.33).

This regularization seems to be natural, as it is a well-known procedure in QFT defined on Minkowski spacetime. There one often encounters pseudo-Gaussian integrals like

$$\int_{\mathbb{R}} \mathrm{d}x\, x^n\,\mathrm{e}^{\,\mathrm{i}\,ax^2} = \sqrt{\frac{\mathrm{i}}{a}}^{\,n+1}\,\Gamma\left(\frac{n+1}{2}\right)\,, \tag{5.55}$$

which have a meaning if regularized in the same way as above. The same integral appears in the scalar product $\langle f_n^+|f_m^-\rangle=\delta_{nm}$ and has to be understood in this way. The regularized $f_n^\pm$ will be denoted in the following by

$$f_n^{(\gamma_\vartheta)}(q)=\left(\frac{\sqrt{\gamma_\vartheta}}{2^n n!\sqrt{\pi}}\right)^{1/2}\mathrm{e}^{-\frac{\gamma_\vartheta}{2}\,q^2}H_n(\sqrt{\gamma_\vartheta}\,q) \tag{5.56}$$

using the compact notation

$$\gamma_\vartheta=\mathrm{e}^{\,\mathrm{i}\,\vartheta}\gamma \tag{5.57}$$

with $\gamma>0$. For $\vartheta\in(-\pi/2,\pi/2)$ they possess an exponential decay due to the Gaussian factor, and are thus Schwartz functions.[3] Apparently, they are eigenfunction of the harmonic oscillator with complex frequency γ_ϑ which is known as *complex harmonic oscillator*. In appendix C.2 we show that their linear span is dense in $L^2(\mathbb{R})$. But the occurring sums are not convergent with respect to L^2-norm and thus do not build a Riesz basis [Dav99, DK04]. Its general applicability will be scrutinized in section 6.2.

To summarize, we have (at least) two different concepts at our disposal to treat the Minkowskian LS-covariant models. There is the continuous approach based on the eigenfunctions $\chi_s^{\mathcal{E}}$ and the matrix approach using the regularized functions $f_n^{(\gamma_\vartheta)}$. The eigenfunctions of the wave operators are given by

$$\chi_{ss'}^{\mathcal{E}\mathcal{E}'}(\boldsymbol{x}) := C_\chi \mathsf{W}\left[|\chi_s^{\mathcal{E}}\rangle\langle\chi_{s'}^{\mathcal{E}'}|\right](\boldsymbol{x}) \tag{5.58}$$

$$f_{mn}^{(2\gamma_\vartheta)}(\boldsymbol{x}) := C_f \mathsf{W}\left[|f_m^{(\gamma_\vartheta)}\rangle\langle f_n^{(\gamma_{-\vartheta})}|\right](\boldsymbol{x}) \tag{5.59}$$

with some normalization constants C_χ and C_f, where the (bi-)orthogonality of $\chi_s^{\mathcal{E}}$ and $f_n^{(\gamma_\vartheta)}$ will ensure the simplification of the $\phi^{\star 4}$ through (2.14) analog to the Euclidean Landau functions. In the next section we will show how to implement the regularized matrix basis.

[3]This is in contradistinction to the original functions $f_n^{(\pm)}=\lim_{\vartheta\to\pm\pi/2} f_n^{(\gamma_\vartheta)}$, whose modulus increases polynomially. They are tempered distributions.

5.4 LS-Covariant NCQFT and the ϑ-Regularization

We have seen that a regularization is needed to endow the functions $f_n^{(\pm)}$ with nice properties. Since these regularized functions, or rather their Wigner transformed counterparts (5.59), will not diagonalize the free actions we are concerned with, it raises the question how to exploit the regularized functions to find a matrix representation for the LS-covariant models on Minkowski spacetime? The answer is that we have to regularize the *action* anyway to define the corresponding quantum field theory. In section 2.2.3 we enhanced the action by an additional term $\mathrm{i}\,\epsilon \int \phi^2$, which ensured the correct asymptotic decay of the integrand within the generating functional at $|\phi| \to \infty$ and at the same time imposed causality. We will now introduce a regularization of the model, which will be called ϑ-regularization and corresponds to the regularization of the matrix function above. In case of vanishing background field this turns out to be the $\mathrm{i}\,\epsilon$-prescription.

We know that the $f_n^{(\gamma_\vartheta)}$ are analytical continuations of the harmonic oscillator functions. So a natural guess for a corresponding generalization of the *free* LS-covariant models is

$$\mathrm{i}\,\mathcal{S}_0[\vartheta] = \mathrm{i}\,\sin(\vartheta)\mathcal{S}_0^M - \cos(\vartheta)\mathcal{S}_0^E \tag{5.60}$$

where $\mathcal{S}_0^M$ stands for the Minkowskian version and $\mathcal{S}_0^E$ for its Euclidean counterpart. Obviously this action relates both signatures, with $\vartheta = 0$ corresponding to the Euclidean and $\vartheta = \pm\pi/2$ to the Minkowskian case. The combinations of the wave operators showing up in $\mathcal{S}_0[\vartheta]$ for the different models are given by

$$\begin{aligned} \mathrm{e}^{-\mathrm{i}\,\vartheta}\mathsf{K}^2(\vartheta) &:= \cos(\vartheta)\mathsf{K}_i^2 - \mathrm{i}\,\sin(\vartheta)\mathsf{K}_\mu^2\,, \\ \mathrm{e}^{-\mathrm{i}\,\vartheta}\tilde{\mathsf{K}}^2(\vartheta) &:= \cos(\vartheta)\tilde{\mathsf{K}}_i^2 - \mathrm{i}\,\sin(\vartheta)\tilde{\mathsf{K}}_\mu^2\,, \end{aligned} \tag{5.61}$$

where the phase factor has been factored out such that

$$\begin{aligned} \mathsf{K}^2(\pm\pi/2) = \mathsf{K}_\mu^2 \quad &, \quad \tilde{\mathsf{K}}^2(\pm\pi/2) = \tilde{\mathsf{K}}_\mu^2\,, \\ \mathsf{K}^2(0) = \mathsf{K}_i^2 \quad &, \quad \tilde{\mathsf{K}}^2(0) = \tilde{\mathsf{K}}_i^2\,. \end{aligned} \tag{5.62}$$

As will be shown below, the wave operators $\mathsf{K}(\vartheta)$ and $\tilde{\mathsf{K}}(\vartheta)$ have discrete spectra with eigenfunctions given by

$$f_{m_1 n_1}^{(E_\vartheta)}(\boldsymbol{x}_1) f_{m_2 n_2}^{(B_2)}(\boldsymbol{x}_2) \cdots f_{m_n n_n}^{(B_n)}(\boldsymbol{x}_n) \tag{5.63}$$

with $\boldsymbol{x}_k = (x^{2k-2}, x^{2k-1})$, $f_{m_k n_k}^{(B_k)}(\boldsymbol{x}_k)$ the usual Landau functions and $f_{m_1 n_1}^{(E_\vartheta)}(\boldsymbol{x}_1)$ given by (5.59). The quantum field theory is defined by the generating functional

$$Z[J] = \lim_{\vartheta \to \pm\pi/2} \int \mathcal{D}\phi\, \exp\left(\mathrm{i}\,\sin(\vartheta)\mathcal{S}_0^M - \cos(\vartheta)\mathcal{S}_0^E + \mathrm{i}\,\mathcal{S}_{\mathsf{int}} + \int J\phi \right) \tag{5.64}$$

for the GW model and analogously for the complex LSZ model, where one of the two options $\vartheta \to \pm\pi/2$ has to be chosen. The free action in the exponent of (5.64) can be expanded in the ϑ-regularized matrix basis. Remember that $\mathcal{S}_0^{\mathrm{E}}$ is a positive functional on the fields, thus for $\vartheta \neq 0$ this can be interpreted as a path integral in Minkowski space with an additional convergence factor, corresponding to the $-\int \epsilon\phi^2$ term in the free case (2.53). Not really surprising, in the limit $E \to 0$ the modification (5.60) turns out to be the $\mathrm{i}\,\epsilon$ prescription of the free case:

$$\sigma\,\mathrm{e}^{-\mathrm{i}\,\vartheta}\mathsf{K}^2(\vartheta) + (1-\sigma)\,\mathrm{e}^{-\mathrm{i}\,\vartheta}\tilde{\mathsf{K}}(\vartheta) + \mathrm{e}^{\,\mathrm{i}\,\vartheta}\mu^2 \xrightarrow{E\to 0} -\mathrm{e}^{-\mathrm{i}\,\vartheta}\partial_0^2 - \mathrm{e}^{\,\mathrm{i}\,\vartheta}\partial_i^2 + \mathrm{e}^{\,\mathrm{i}\,\vartheta}\mu^2\,, \tag{5.65}$$

which holds for all σ. The ϑ-regularization is hence a generalization of the $\mathrm{i}\,\epsilon$-prescription to the external electromagnetic field case! One is thus tempted to interpret the two different models near $+\pi/2$ is $-\pi/2$ analogously to the situation in the free case. There, flipping the sign in the exponential of the path integral interchanges the particle and anti-particle description, by interchanging the Feynman- and *Dyson-propagator* (also known as *anti-causal propagator*) which correspond to different ways of circumventing the poles. It will be shown in section 7.1.1 that this interpretation indeed holds for $E \neq 0$. Without restriction of generality we will choose in the following always $\vartheta > 0$ and define $\vartheta = \pi/2 - \epsilon > 0$ for a small $\epsilon > 0$. Denoting

$$(\mathsf{K}_\mu^2 - \mu^2)_\epsilon \quad := \quad \mathrm{e}^{\,\mathrm{i}\,\epsilon}\,\mathsf{K}^2(\pi/2 - \epsilon) - \mathrm{e}^{-\mathrm{i}\,\epsilon}\mu^2 \tag{5.66}$$

$$(\tilde{\mathsf{K}}_\mu^2 - \mu^2)_\epsilon \quad := \quad \mathrm{e}^{\,\mathrm{i}\,\epsilon}\,\tilde{\mathsf{K}}^2(\pi/2 - \epsilon) - \mathrm{e}^{-\mathrm{i}\,\epsilon}\mu^2 \tag{5.67}$$

the regularized LSZ model is defined by the classical action

$$\begin{aligned} \mathcal{S}_{\mathsf{LSZ}}^{(\epsilon)} = \int_x \phi^*(\boldsymbol{x}) \Big(\sigma(\mathsf{K}_\mu^2 - \mu^2)_\epsilon + (1-\sigma)(\tilde{\mathsf{K}}_\mu^2 - \mu^2)_\epsilon\Big)\, \phi(\boldsymbol{x}) \\ - g\left(\alpha \int_x (\phi^* \star \phi \star \phi^* \star \phi)(\boldsymbol{x}) + \beta \int_x (\phi^* \star \phi^* \star \phi \star \phi)(\boldsymbol{x})\right) \end{aligned} \tag{5.68}$$

and the regularized GW model by

$$\mathcal{S}_{\mathsf{GW}}^{(\epsilon)} = \int_x \frac{1}{2}\phi(\boldsymbol{x}) \left(\frac{1}{2}(\mathsf{K}_\mu^2 - \mu^2)_\epsilon + \frac{1}{2}(\tilde{\mathsf{K}}_\mu^2 - \mu^2)_\epsilon\right) \phi(\boldsymbol{x}) - g \int_x (\phi \star \phi \star \phi \star \phi)\,(\boldsymbol{x})\,. \tag{5.69}$$

What remains is to show that these actions get indeed diagonalized by the functions (5.63) in some space of functions. This will be shown in the next chapter.

Note that the usual $\mathrm{i}\,\epsilon$-prescription amounts to adding the constant $\mathrm{i}\,\epsilon$ to the continuous spectrum of the wave operators, but leaves its continuous character unaltered. A perturbative quantum theory amenable for the continuous basis approach with functions (5.58) is the generating functional

$$Z[J] = \lim_{\epsilon \to 0^+} \int \mathcal{D}\phi\, \exp\left(\mathrm{i}\,\mathcal{S}_0^M - \epsilon \int \phi^2 + \mathrm{i}\,\mathcal{S}_{\mathsf{int}} + \int J\phi\right)\,. \tag{5.70}$$

We thus have two possible definitions for a generating functional, while it is not obvious that both are equivalent.

The perturbation theory of the Minkowskian LS-covariant NCQFT can be derived quite similar to the usual $\phi^{\star 4}$ theory 2.2.3, where the background field is treated exactly in the Furry representation as in section 3.2. For real fields we write the regularized free actions as

$$\mathcal{S}_0^{(\epsilon)} = \int_x \phi(\boldsymbol{x})\, \mathsf{D}_x^{(\epsilon)}\, \phi(\boldsymbol{x}) \tag{5.71}$$

where $\mathsf{D}_x^{(\epsilon)}$ is the wave operator, which has been regularized in one of two possible ways. The regularization ensures the vanishing of the integrand in the path integrals for $|\phi| \to \infty$ leading to the free generating functional

$$Z_0[J] = \lim_{\epsilon \to 0^+} \exp\left(\frac{\mathrm{i}}{2} \int_x \int_y J(\boldsymbol{x})\, \Delta^{(\epsilon)}(\boldsymbol{x},\boldsymbol{y})\, J(\boldsymbol{y})\right)\,. \tag{5.72}$$

with $\Delta^{(\epsilon)}$ the propagator defined through one of the equations

$$\begin{aligned} \Big(\sigma(\mathsf{K}_\mu^2 - \mu^2)_\epsilon + (1-\sigma)(\tilde{\mathsf{K}}_\mu^2 - \mu^2)_\epsilon\Big)\, \Delta^{(\epsilon)}(\boldsymbol{x},\boldsymbol{y}) = \delta(\boldsymbol{x}-\boldsymbol{y})\,, \\ \Big(\sigma\mathsf{K}_\mu^2 + (1-\sigma)\tilde{\mathsf{K}}_\mu^2 - \mu^2 + \mathrm{i}\,\epsilon\Big)\, \Delta^{(\epsilon)}(\boldsymbol{x},\boldsymbol{y}) = \delta(\boldsymbol{x}-\boldsymbol{y})\,. \end{aligned} \tag{5.73}$$

This is the point where the regularization could make the difference, since it is not clear initially whether these two propagators coincide in the limit $\epsilon \to 0^+$. We will come back to this point in chapter 7. The formal setting of a perturbative analysis of the interacting NCQFTs is now given by

$$Z[J] = \lim_{\epsilon \to 0^+} \mathcal{N} \exp\left[\mathrm{i}\,\mathcal{S}_{\mathsf{int}}\left(\frac{\delta}{\delta J}\right)\right] \exp\left(\frac{\mathrm{i}}{2}\int_x\int_y J(\boldsymbol{x})\, \Delta^{(\epsilon)}(\boldsymbol{x},\boldsymbol{y})\, J(\boldsymbol{y})\right)\,. \tag{5.74}$$

with $\mathcal{S}_{\mathsf{int}}$ the interaction part and $\mathcal{N}$ some normalization constant. For complex fields we get

$$Z[J,J^*] = \lim_{\epsilon \to 0^+} \mathcal{N} \exp\left[\mathrm{i}\,\mathcal{S}_{\mathsf{int}}\left(\frac{\delta}{\delta J^*}, \frac{\delta}{\delta J}\right)\right] \exp\left(\mathrm{i}\int_x\int_y J^*(\boldsymbol{x})\, \Delta^{(\epsilon)}(\boldsymbol{x},\boldsymbol{y})\, J(\boldsymbol{y})\right)\,. \tag{5.75}$$

In the following chapter we will construct the matrix representations of the regularized LS-covariant models.

6 Matrix Model Representation of Minkowskian LS-Covariant NCQFT

In the previous chapter we showed that it is possible to find a matrix representation for the LS-covariant models through a suitable regularization, which has been dubbed *ϑ-regularization* and is an alternative to the usual $\mathrm{i}\,\epsilon$-prescription. In this chapter we will try to nail this matrix representation down. In section 6.1 we will use the Weyl-Wigner transformation to map the eigenvalue problem of the regularized wave operators (5.61) to the *complex* harmonic oscillator. Its spectrum and eigenfunctions are investigated in section 6.2, as well as the possibility to expand functions and distributions in terms of these eigenfunctions. The generalized Landau functions are constructed in section 6.3. Using their Fock space representation, we will finally arrive at the matrix model representation for the two-dimensional classical models in 6.4 and their corresponding quantum theories. The generalization to higher dimensions is illustrated in section 6.5.

6.1 Mapping onto the Complex Harmonic Oscillator

The first step is to find the corresponding Weyl symbols of the generalized operators

$$\begin{aligned} \mathsf{K}^2(\vartheta) &= \mathrm{e}^{\,\mathrm{i}\,\vartheta}\left(\cos(\vartheta)\mathsf{K}_i^2 - \mathrm{i}\,\sin(\vartheta)\mathsf{K}_\mu^2\right)\,, \\ \tilde{\mathsf{K}}^2(\vartheta) &= \mathrm{e}^{\,\mathrm{i}\,\vartheta}\left(\cos(\vartheta)\tilde{\mathsf{K}}_i^2 - \mathrm{i}\,\sin(\vartheta)\tilde{\mathsf{K}}_\mu^2\right)\,, \end{aligned} \tag{6.1}$$

similar to the Euclidean and Minkowskian cases in section 4.1 and 5.2, which again split up into two-dimensional wave operators defined by (3.12), (5.8) and (5.9). In $D = 2n$ dimensions the components $(\mathsf{P}_i^2)_k$ and $(\mathsf{P}_\mu^2)_k$, and likewise $(\tilde{\mathsf{P}}_i^2)_k$ and $(\tilde{\mathsf{P}}_\mu^2)_k$, differ only by a minus sign for $k = 2, \dots, n$ up to a relabeling of the coordinates. We thus find

$$\begin{aligned} \mathsf{K}^2(\vartheta) &= \mathrm{e}^{\,\mathrm{i}\,\vartheta}\left(\cos(\vartheta)(\mathsf{P}_i^2)_1 - \mathrm{i}\,\sin(\vartheta)(\mathsf{P}_\mu^2)_1\right) + \mathrm{e}^{\,2\,\mathrm{i}\,\vartheta}\sum_{k=2}^{n}(\mathsf{P}_i^2)_k\,, \\ \tilde{\mathsf{K}}^2(\vartheta) &= \mathrm{e}^{\,\mathrm{i}\,\vartheta}\left(\cos(\vartheta)(\tilde{\mathsf{P}}_i^2)_1 - \mathrm{i}\,\sin(\vartheta)(\tilde{\mathsf{P}}_\mu^2)_1\right) + \mathrm{e}^{\,2\,\mathrm{i}\,\vartheta}\sum_{k=2}^{n}(\tilde{\mathsf{P}}_i^2)_k\,. \end{aligned} \tag{6.2}$$

What remains is to find the eigenfunctions of the remaining parts of the wave operators. We denote the $k = 1$ part as

$$\begin{aligned} \mathsf{P}^2(\vartheta) &= \mathrm{e}^{\,\mathrm{i}\,\vartheta}\left(\cos(\vartheta)(\mathsf{P}_i^2)_1 - \mathrm{i}\,\sin(\vartheta)(\mathsf{P}_\mu^2)_1\right)\,, \\ \tilde{\mathsf{P}}^2(\vartheta) &= \mathrm{e}^{\,\mathrm{i}\,\vartheta}\left(\cos(\vartheta)(\tilde{\mathsf{P}}_i^2)_1 - \mathrm{i}\,\sin(\vartheta)(\tilde{\mathsf{P}}_\mu^2)_1\right)\,. \end{aligned} \tag{6.3}$$

Using (5.13), one easily confirms that

$$\begin{aligned} \mathsf{P}^2(\vartheta)\,f(\boldsymbol{x}) &= H(\vartheta)\star_{2/|E|} f(\boldsymbol{x})\,, \\ \tilde{\mathsf{P}}^2(\vartheta)\,f(\boldsymbol{x}) &= f(\boldsymbol{x})\star_{2/|E|} H(\vartheta)\,, \end{aligned} \tag{6.4}$$

with

$$H(\vartheta) := E^2(x^2 + \mathrm{e}^{\,2\,\mathrm{i}\,\vartheta}t^2)\,. \tag{6.5}$$

Allocating to each function $H(\vartheta)$ a Weyl symbol $\hat{\mathbf{H}}(\vartheta)$ we find

$$\begin{aligned} \hat{\mathbf{H}}(\vartheta) &= \frac{1}{2}\left(\hat{\mathcal{W}}\left[\sqrt{2}Ex\right]^2 + \mathrm{e}^{\,2\,\mathrm{i}\,\vartheta}\hat{\mathcal{W}}\left[\sqrt{2}Et\right]^2\right) \\ &= \frac{1}{2}\left(\hat{\mathbf{p}}^2 + \mathrm{e}^{\,2\,\mathrm{i}\,\vartheta}\hat{\mathbf{q}}^2\right)\,, \end{aligned} \tag{6.6}$$

where the symbols $\hat{\mathcal{W}}\left[\sqrt{2}Ex\right] = \hat{\mathbf{p}}$ and $\hat{\mathcal{W}}\left[\sqrt{2}Et\right] = \hat{\mathbf{q}}$ obey the Heisenberg algebra

$$[\hat{\mathbf{q}}, \hat{\mathbf{p}}] = 2E^2[t, x]_{\star_{2/E}} = \mathrm{i}\, 4E\,. \tag{6.7}$$

The operators $\hat{\mathbf{H}}(\vartheta)$ for $\vartheta \in (-\pi/2, \pi/2)$ are known as *complex harmonic oscillators*, and the eigenvalue equations of our original operators are related to their correspondents on the Weyl side by

$$\begin{aligned} \mathsf{P}^2(\vartheta) f_{mn}^{(E_\vartheta)}(\boldsymbol{x}) &= \mathsf{W}\left[\hat{\mathbf{H}}(\vartheta)\hat{\mathbf{f}}_{mn}^{(E_\vartheta)}\right](\boldsymbol{x}) = \lambda_{mn}^{(E_\vartheta)} f_{mn}^{(E_\vartheta)}(\boldsymbol{x}) \\ \tilde{\mathsf{P}}^2(\vartheta) f_{mn}^{(E_\vartheta)}(\boldsymbol{x}) &= \mathsf{W}\left[\hat{\mathbf{f}}_{mn}^{(E_\vartheta)}\hat{\mathbf{H}}(\vartheta)\right](\boldsymbol{x}) = \tilde{\lambda}_{mn}^{(E_\vartheta)} f_{mn}^{(E_\vartheta)}(\boldsymbol{x}) \end{aligned} \tag{6.8}$$

with $\hat{\mathbf{f}}_{mn}^{(E_\vartheta)} = \hat{\mathcal{W}}[f_{mn}^{(E_\vartheta)}]$. The spectrum of $\hat{\mathbf{H}}(\vartheta)$ and its eigenfunctions $\hat{\mathbf{f}}_{mn}^{(E_\vartheta)}$ will be investigated in the next section. The eigenvalues will turn out to depend on E and ϑ only through the combination

$$E\,\mathrm{e}^{\,\mathrm{i}\,\vartheta} := E_\vartheta\,, \tag{6.9}$$

whereas the eigenfunctions are tensor products of two generalized oscillator functions of frequency $E_\vartheta/2$, which explains the (E_ϑ)-superscript of the functions. The simultaneous eigenfunctions of $\mathsf{P}^2(\vartheta)$ and $\tilde{\mathsf{P}}^2(\vartheta)$ can afterwards be achieved with help of the Wigner transformation

$$f_{mn}^{(E_\vartheta)}(\boldsymbol{x}) = \mathsf{W}[\hat{\mathbf{f}}_{mn}^{(E_\vartheta)}](\boldsymbol{x})\,. \tag{6.10}$$

The extension to the full D-dimensional case will be given in section 6.5.

6.2 Generalized Oscillator Basis

In this section investigate the complex harmonic oscillator $\hat{\mathbf{H}}(\vartheta)$, which turns out to have a discrete spectrum resembling the harmonic oscillator spectrum rotated into the complex plane by a factor $\mathrm{e}^{\,\mathrm{i}\,\vartheta}$, and whose eigenfunctions are found to be the regularized harmonic oscillator functions $f_n^{(\gamma_\vartheta)}$ of section 5.3. In addition, the general applicability of the generalized oscillator basis is scrutinized.

The complex harmonic oscillator in a representation independent form is given by

$$\hat{\mathbf{H}}_{\text{ho}} = \frac{1}{2}(\hat{\mathbf{p}}^2 + \mathrm{e}^{\,2\,\mathrm{i}\,\vartheta}\hat{\mathbf{q}}^2) \tag{6.11}$$

with commutation relation

$$[\hat{\mathbf{q}}, \hat{\mathbf{p}}] = \mathrm{i}\, 4E \tag{6.12}$$

and positive real frequency $E \in \mathbb{R}_+$. Since $\hat{\mathbf{q}} = \hat{\mathcal{W}}\left[\sqrt{2}Et\right]$, it is natural to work in a representation such that

$$\langle q'|\hat{\mathbf{q}}|q\rangle = \sqrt{2}Eq\,\langle q'|q\rangle \quad \Rightarrow \langle q'|\hat{\mathbf{p}}|q\rangle = -\,\mathrm{i}\,\frac{\partial}{\partial q/\sqrt{8}}\,\langle q'|q\rangle \tag{6.13}$$

thus

$$\langle q'|\hat{\mathbf{H}}(\vartheta)|q\rangle \quad = \quad 4\left(-\partial_q + \gamma_\vartheta^2\, q^2\right)\langle q'|q\rangle \tag{6.14}$$

with the condensed notation

$$\gamma_\vartheta = \mathrm{e}^{\,\mathrm{i}\,\vartheta}\gamma \quad \text{with} \quad \gamma = E/2 \in \mathbb{R}_+\,. \tag{6.15}$$

Firstly note that the equation

$$4(-\partial_q^2 + \gamma_\vartheta^2 q^2)\phi_n(q) = 8\gamma_\vartheta\left(n + \frac{1}{2}\right)\phi_n(q) \tag{6.16}$$

is fulfilled even for complex γ_ϑ, if $\phi_n(q)$ is the oscillator function (4.16) with γ_ϑ substituted for γ. We define the *generalized harmonic oscillator functions*

$$f_n^{(\gamma_\vartheta)}(q) = \left(\frac{\sqrt{\gamma_\vartheta}}{2^n n! \sqrt{\pi}}\right)^{1/2} \mathrm{e}^{-\frac{\gamma_\vartheta}{2}q^2} H_n(\sqrt{\gamma_\vartheta} q) \tag{6.17}$$

as a generalization of the ϕ_n to complex frequencies, which coincide with the functions found in section 5.3. These possess an exponential decay and are thus Schwartz functions for $|\vartheta| < \pi/2$. We expect that by continuity, for $|\vartheta|$ small enough, the eigenvalues of the complex harmonic oscillator are given by the set

$$\{8\gamma_\vartheta\,(n+1/2)\,,\, n \in \mathbb{N}\}\,. \tag{6.18}$$

In fact, the values (6.18) are indeed the eigenvalues of $\hat{\mathbf{H}}(\vartheta)$ for $|\vartheta| < \pi/2$ [Dav99].

The generalized harmonic oscillator functions (6.17) are not orthogonal, and thus do not serve as a usual Hilbert space basis for $S(\mathbb{R})$. But together with its complex conjugated functions and for $\mathfrak{Re}(\gamma_\vartheta) > 0$, they constitute a bi-orthogonal system with respect to the L^2-norm. This means the two sets of functions $(f_n^{(\gamma_\vartheta)})_{n\in\mathbb{N}}$ and $(f_n^{(\gamma_{-\vartheta})})_{n\in\mathbb{N}}$ with nonzero γ_ϑ and $\mathfrak{Re}(\gamma_\vartheta) > 0$ fulfill

$$\langle f_n^{(\gamma_{-\vartheta})} | f_m^{(\gamma_\vartheta)} \rangle = \int_{-\infty}^{\infty} \mathrm{d}q\, f_n^{(\gamma_\vartheta)}(q)\, f_m^{(\gamma_\vartheta)}(q) = \delta_{nm}\,. \tag{6.19}$$

which follows immediately from the orthogonality of the Hermite functions by a deformation of the integration contour to a straight line from $-\infty\,\mathrm{e}^{\,\mathrm{i}\,\vartheta}$ to $+\infty\,\mathrm{e}^{\,\mathrm{i}\,\vartheta}$. This is possible due to the factor $\mathrm{e}^{-\frac{\gamma_\vartheta}{2}q^2}$ in the integrand, ensuring an exponential decay for $\mathfrak{Re}(\gamma_\vartheta) > 0$. In addition their linear span is dense in $L^2(\mathbb{R})$, which means that every square-integrable function can be approximated pointwise by a linear combination of these functions. This is shown in appendix D. To ensure the applicability to arbitrary quantum field theories, however, one has also to be able to deal with scalar products and distributions. In the following we will first briefly explain how things work out in the usual oscillator basis $\phi_n(q)$ with positive frequency $\gamma \in \mathbb{R}_+$. Afterwards we will present preliminary results concerning the generalized oscillator basis.

The usual oscillator basis provide a convenient tool in the investigation of tempered distributions and similar objects. Characterizations of standard classes of functions, as Schwartz space $S(\mathbb{R})$ and its dual $S'(\mathbb{R})$ and many others are easily given in terms of their expansion coefficients with respect to the oscillator functions [Sim70], which in the following will be called the *Hermite coefficients*. Since the issue of how to implement NCQFT into a mathematically rigorous formalism has still to be clarified, see e.g. [BN04, Sol07b, Sol07a, CMTV08, Sol09, Sol10], we will only discuss the expansion of several spaces in terms of the oscillator basis and its generalization. The characterization of Schwartz functions is as follows. For a function $\varphi(x) \in S(\mathbb{R})$ with Hermite coefficients

$$\varphi_n = \int_{\mathbb{R}} \mathrm{d}q\, \phi_n(q)\, \varphi(q)\,, \tag{6.20}$$

one finds [Sim70]

$$\|\varphi\|_k^2 := \sum_n |\varphi_n|^2 (n+1)^k < \infty \tag{6.21}$$

for every $k \in \mathbb{N}$. If on the other hand $\|\psi\|_k < \infty$ for all k, then $\sum_n \psi_n \phi_n(x)$ converges in the Schwartz topology to a function in $S(\mathbb{R})$, establishing an isomorphism between the Schwartz space and the space of fast falling sequences. Moreover convergence in the topology of $S(\mathbb{R})$ is equivalent to convergence of their Hermite coefficients with respect to the infinite set of norms $\|\cdot\|_k$ for $k \in \mathbb{N}$.

Now suppose that $T \in \mathcal{S}'(\mathbb{R})$ is a tempered distribution with $T_n = T(\phi_n)$ denoting its Hermite coefficients. Then $|T_n| \le C(1+n)^k$ for some C and k and

$$T(\varphi) = \sum_{n=0}^{\infty} T_n \varphi_n\,, \tag{6.22}$$

for any $\varphi \in \mathcal{S}(\mathbb{R})$ with Hermite coefficients φ_n. Conversely, if $T_n \le C(1+n)^k$ for some k and all n, then $\varphi \mapsto \sum_n T_n \varphi_n$ defines a tempered distribution. By duality the usual oscillator basis thus provides a mean to deal with tempered distributions.

The extension to larger spaces of distributions, like the space $\mathcal{S}_\alpha^\alpha(\mathbb{R})'$ with $\alpha \geq 1/2$, which is the dual of the *Gel'fand-Shilov space* $\mathcal{S}_\alpha^\alpha(\mathbb{R})$, is also possible in the same manner. The space $\mathcal{S}_\alpha^\alpha(\mathbb{R})$ is dense in the Schwartz space, closed under Fourier transformation and the star-product, which makes them to an appropriate test function space for noncommutative field theories. For a short introduction see appendix C.1. In [LCP07] it has been shown that its elements are exactly those fast-falling functions φ, whose Hermite coefficients $\varphi_n = \langle \phi_n | \varphi \rangle$ fulfill the condition

$$\sum_{n=0}^{\infty} |\varphi_n|^2 \, \mathrm{e}^{\, n^{\frac{1}{2\alpha}} \omega} < 0 \, . \tag{6.23}$$

for some constant $\omega > 0$. Its dual space $\mathcal{S}_\alpha^\alpha(\mathbb{R})'$ consists of those distributions T, whose Hermite coefficients $T_n = T(\phi_n)$ satisfy

$$|T_n| \leq \mathrm{e}^{\, n^{\frac{1}{2\alpha}} \omega} \tag{6.24}$$

for all $\omega > 0$, and $T(\varphi)$ for every $\varphi \in \mathcal{S}_\alpha^\alpha(\mathbb{R})$ has the representation

$$T(\varphi) = \sum_{n=0}^{\infty} T_n \varphi_n \, . \tag{6.25}$$

Conversely, for any sequence $(T_n)_{n\in\mathbb{N}}$ satisfying (6.24) for all $\omega > 0$, then $\varphi \to \sum_n T_n \varphi_n$ defines an element of $\mathcal{S}_\alpha^\alpha(\mathbb{R})'$.

The question is, if a similar characterization holds if we continue γ into the complex plane, thus for the expansion in generalized oscillator functions. Analogously to the ordinary case described above, we would like to define the action of a tempered distribution $T \in S'(\mathbb{R})$ on test functions φ by

$$T(\varphi) = \sum_{n=0}^{\infty} T_n^{(\gamma_\vartheta)} \varphi_n^{(\gamma_\vartheta)} \, , \tag{6.26}$$

where

$$\varphi_n^{(\gamma_\vartheta)} \quad = \quad \langle f_n^{(\gamma_{-\vartheta})} | \varphi \rangle \, . \tag{6.27}$$

and

$$T_n^{(\gamma_\vartheta)} \quad = \quad T(f_n^{(\gamma_\vartheta)}) \tag{6.28}$$

for nonzero $\gamma_\vartheta \in \mathbb{C}$ with $\mathfrak{Re}(\gamma_\vartheta) > 0$. The generalized Hermite coefficients $T_n^{(\gamma_\vartheta)}$ exists for every tempered distribution since $f_n^{(\gamma_\vartheta)} \in \mathcal{S}(\mathbb{R})$. However, it is not clear for which functions φ the series (6.26) is well-defined.

Concerning this question we only have partial results. Note that for any tempered distribution (or Schwartz function) ψ we can formally switch between the usual Hermite coefficients, defined for nonzero frequency $\gamma \in \mathbb{R}_+$, and rotated Hermite coefficients with frequency γ_ϑ through

$$\psi_n^{(\gamma_\vartheta)} = \sum_{m=0}^{\infty} h_{nm}^{(\vartheta)} \psi_m \, , \tag{6.29}$$

where

$$h_{nm}^{(\vartheta)} = \int_{\mathbb{R}} \mathrm{d}q \, f_n^{(\gamma_\vartheta)}(q) \, \phi_m(q) \tag{6.30}$$

is the *transition matrix*. This follows from equation (6.22) for $\varphi = f_n^{(\gamma)}$ and $T = \psi$. In appendix B.3 we show, that for arbitrary nonzero, distinct $\beta, \gamma \in \mathbb{C}$ with $\mathfrak{Re}(\gamma + \beta) > 0$ the general transition matrix

$$h_{nm}^{(\gamma,\beta)} := \int_{\mathbb{R}} \mathrm{d}q \, f_n^{(\gamma)}(q) \, f_m^{(\beta)}(q) \tag{6.31}$$

has the following asymptotic behavior for given m and large n:

$$h_{nm}^{(\gamma,\beta)} \overset{n\to\infty}{\sim} n^{-1/2} \left| \frac{\beta-\gamma}{\beta+\gamma} \right|^n . \tag{6.32}$$

We see that the transition matrix has an exponential decay if the angle between β and γ is less than $\pi/2$. To answer the question whether there is are functions such that the expansion (6.26) of tempered distributions is allowed, we have to find the asymptotics of the corresponding generalized Hermite functions. This can be done using the transition matrix and relation (6.29), since the asymptotics of the usual Hermite coefficients are known.

A space which is computationally feasible is the Gel'fand-Shilov space of type $\mathcal{S}_\alpha^\alpha(\mathbb{R}) \subset \mathcal{S}(\mathbb{R})$ with $\alpha = 1/2$. In appendix C.2 we show that for this case the corresponding generalized Hermite coefficients $\varphi_n^{(\gamma_\vartheta)}$ have the following asymptotic upper bound for large n

$$|\varphi_n^{(\gamma_\vartheta)}| \lesssim \frac{1+\mathrm{e}^{2\omega} r}{\mathrm{e}^{2\omega}-r} . \tag{6.33}$$

with

$$r = |\tan(\vartheta/2)| . \tag{6.34}$$

Thus for a given $r \in [0,1]$ there is a lower bound ω_0 given by

$$\omega_0 = \frac{1}{2} \ln \left(\frac{1+r}{1-r} \right) \tag{6.35}$$

such that every Gel'fand-Shilov function with $\omega > \omega_0$ has an exponential decay. However, this is not a precise lower bound, since we used a rough estimation to obtain this result. The actual asymptotics for those functions might be better. For fixed ϑ there is thus a space of functions which might serve as test function space. If we allow r to become arbitrary close to 1, according to the asymptotics of the transition matrix (6.32) we have to restrict to those functions whose usual Hermite coefficients decay faster than $\mathrm{e}^{-n\omega}$ for every ω. The space of functions obeying this condition is space made up of all finite linear combinations of harmonic oscillators, thus the space spanned by the ϕ_n. But this is obvious, since every finite linear combination $\sum_{m=0}^N \phi_m(q) a_m$ gives rise to a function in the rotated oscillator basis with generalized Hermite coefficients

$$a_n^{(\vartheta)} = \sum_{m=0}^{N} h_{nm}^{(\vartheta)} a_m , \tag{6.36}$$

which, according to the asymptotics of the transition matrix $h_{nm}^{(\vartheta)}$ given by (6.32), have an exponential decay for $|\vartheta| < \pi/2$ in the limit $n \to \infty$. This space is obviously dense in $L^2(\mathbb{R})$ pointwise.

Using the same methods, analog results may be derived for tempered distributions giving exponential divergences

$$|T_n^{(\gamma_\vartheta)}| \sim (n+1)^q \left(\frac{1+r}{1-r} \right)^{n/2} \tag{6.37}$$

for some $q > 0$, which have been derived in appendix C.3. Using these upper bounds, one can find sufficient conditions on the test functions such that the sum (6.26) converges. In order to get a decay which damps the divergence of (6.37) we find the condition

$$2\omega > \frac{2-(1-r)^2}{2-(1+r)^2} . \tag{6.38}$$

This has only finite solutions ω for $r < \sqrt{2}-1$ or equivalently $\vartheta < \pi/4$, ruling out test functions made up of an infinite linear combination of oscillator functions. Again, we have to emphasize that these are rough estimates and the actual decay behavior might be much better.

The question for which spaces of functions this generalized oscillator basis makes sense is thus still open and will be left for future work. For a specific theory, however, one only needs the asymptotics of the matrix version of the corresponding propagator, to ensure the convergence of the sums in Feynman diagrams and to proceed with the renormalization program. We will come back to this aspects in chapter 8 and comment on the applicability to LS-covariant theories. In the forthcoming chapters, we will use the matrix basis to derive the propagators of the various theories and find that they coincide with the position space propagators in all those cases, where results are already known in the literature. In appendix F the one-loop effective action of the Klein-Gordon theory in a constant electric field is calculated with help of the matrix basis and also coincides with the known results. By picking up the regularization scheme imposed on the position space propagator in the Euclidean case, which effectively cuts off the matrix summations at some finite N, the occurring Feynman diagrams of the ϑ-regularized LS-covariant theories are well-defined and LS-covariant.

6.3 Generalized Landau functions

In the following, we go back to the Wigner side, by constructing the generalized Landau functions $f_{mn}^{(E_\vartheta)}$, defined by (4.19) through Wigner distribution of the tensor product of two generalized oscillator functions. We will derive a “ladder operator”-construction, which allows us to obtain the matrix model representation of the LS-covariant models. Temporarily we set $\theta = 2/E$ and thus $\star = \star_{2/E}$.

We can use a similar construction as in the Euclidean case in section 4.2 by relating the ordinary to the complex harmonic oscillator functions using *complex scaling* methods. Introducing the Hermitian scaling operator

$$\hat{\mathbf{V}}(\vartheta) = \exp\left(-\frac{\vartheta}{4\gamma}(\hat{\mathbf{p}}\hat{\mathbf{q}} + \hat{\mathbf{q}}\hat{\mathbf{p}})\right) \tag{6.39}$$

and using

$$\mathrm{e}^{X} Y \mathrm{e}^{-X} = \mathrm{e}^{ad\, X} Y = Y + [X,Y] + \frac{1}{2!}[X,[X,Y]] + \dots \tag{6.40}$$

we see that

$$\begin{aligned} \hat{\mathbf{V}}(\vartheta)\, \hat{\mathbf{q}}\, \hat{\mathbf{V}}(\vartheta)^{-1} &= \mathrm{e}^{\,\mathrm{i}\frac{\vartheta}{2}} \hat{\mathbf{q}}\,, \\ \hat{\mathbf{V}}(\vartheta)\, \hat{\mathbf{p}}\, \hat{\mathbf{V}}(\vartheta)^{-1} &= \mathrm{e}^{-\mathrm{i}\frac{\vartheta}{2}} \hat{\mathbf{p}}\,. \end{aligned} \tag{6.41}$$

The complex harmonic oscillator is thus related to the ordinary one by

$$\begin{aligned} \hat{\mathbf{H}}(\vartheta) &:= \mathrm{e}^{\,\mathrm{i}\,\vartheta}\, \hat{\mathbf{V}}(\vartheta)\, \hat{\mathbf{H}}_{\mathrm{ho}}\, \hat{\mathbf{V}}(\vartheta)^{-1} \\ &= \frac{1}{2}\left(\hat{\mathbf{p}}^2 + \mathrm{e}^{\,2\,\mathrm{i}\,\vartheta} \hat{\mathbf{q}}^2\right)\,, \end{aligned} \tag{6.42}$$

while the generalized eigenfunctions can now easily obtained by the oscillator functions $|\phi_n\rangle$, where $\langle q|\phi_n\rangle = \phi_n(q)$ as in (4.16), by noting that

$$\hat{\mathbf{H}}(\vartheta)\, \hat{\mathbf{V}}(\vartheta)|\phi_n\rangle = \mathrm{e}^{\,\mathrm{i}\,\vartheta} \hat{\mathbf{V}}(\vartheta)\, \hat{\mathbf{H}}_{\mathrm{ho}}\, |\phi_n\rangle = \mathrm{e}^{\,\mathrm{i}\,\vartheta} 8\gamma\,(n+1/2)\, \hat{\mathbf{V}}(\vartheta)\, |\phi_n\rangle \tag{6.43}$$

and the corresponding eigenvectors are related to the oscillator wave functions by

$$f_n^{(\gamma_\vartheta)}(q) = \langle q|f_n^{(\gamma_\vartheta)}\rangle := \langle q|\hat{\mathbf{V}}(\vartheta)|\phi_n\rangle = \mathrm{e}^{\,\mathrm{i}\,\vartheta/4} \phi_n\left(\mathrm{e}^{\,\mathrm{i}\,\vartheta/2}\, q\right)\,. \tag{6.44}$$

From here on we can continue deriving the corresponding results for the generalized oscillator functions similar to the Euclidean case, with generalized Landau function given by

$$f_{mn}^{(E_\vartheta)}(\boldsymbol{x}) = \sqrt{\frac{E}{4\pi}}\, \mathsf{W}\left[\hat{\mathbf{V}}(\vartheta)|\phi_m\rangle\langle\phi_n|\hat{\mathbf{V}}(-\vartheta)\right](\boldsymbol{x})\,. \tag{6.45}$$

where normalization constant has been chosen such that again

$$\int \mathrm{d}\boldsymbol{x}\, f_{mn}^{(E_\vartheta)}(\boldsymbol{x}) = \sqrt{\frac{4\pi}{E}}\delta_{mn}\,. \tag{6.46}$$

Using the explicit representation for the Wigner transformation (2.10) we see that complex conjugation yields

$$f_{mn}^{(E_\vartheta)}(\boldsymbol{x})^* \quad = \quad \sqrt{\frac{E}{4\pi}} \int \mathrm{d}k \; \mathrm{e}^{-\,\mathrm{i}\,\frac{E}{2}kx} \langle t+k/2|\hat{\mathbf{V}}(-\vartheta)|\phi_n\rangle\langle\phi_m|\hat{\mathbf{V}}(\vartheta)|t-k/2\rangle = f_{nm}^{(E_{-\vartheta})}(\boldsymbol{x}) \tag{6.47}$$

and the *projector property* takes the form

$$\left(f_{mn}^{(E_\vartheta)} \star f_{k\ell}^{(E_\vartheta)}\right)(\boldsymbol{x}) = \frac{E}{4\pi}\mathsf{W}\left[\hat{\mathbf{V}}(\vartheta)|\phi_m\rangle\langle\phi_n|\phi_k\rangle\langle\phi_\ell|\hat{\mathbf{V}}(-\vartheta)\right](\boldsymbol{x}) = \sqrt{\frac{E}{4\pi}}\,\delta_{nk} f_{m\ell}^{(E_\vartheta)}(\boldsymbol{x})\,. \tag{6.48}$$

Together with the normalization condition this implies the bi-orthogonality of the generalized Landau functions with respect to the L^2 scalar product

$$\begin{aligned}
\langle f_{mn}^{(E_\vartheta)}|f_{k\ell}^{(E_{-\vartheta})}\rangle &= \int \mathrm{d}\boldsymbol{x}\, f_{nm}^{(E_{-\vartheta})}(\boldsymbol{x})\, f_{k\ell}^{(E_{-\vartheta})}(\boldsymbol{x}) \\
&= \int \mathrm{d}\boldsymbol{x}\, \left(f_{nm}^{(E_{-\vartheta})} \star f_{k\ell}^{(E_{-\vartheta})}\right)(\boldsymbol{x}) \\
&= \sqrt{\frac{E}{4\pi}} \int \mathrm{d}\boldsymbol{x}\, \delta_{mk} f_{n\ell}^{(E_{-\vartheta})}(\boldsymbol{x}) \\
&= \delta_{mk}\delta_{n\ell}\,.
\end{aligned} \tag{6.49}$$

The explicit expressions of the matrix basis functions are given by

Theorem 6.1. *The generalized Landau functions $f_{mn}^{(E_\vartheta)}(\boldsymbol{x})$ with $m,n \in \mathbb{N}_0$ are given by*

$$\begin{aligned}
f_{mn}^{(E_\vartheta)}(t,x) &= (-1)^{\min(m,n)}\sqrt{\frac{E}{\pi}}\sqrt{\frac{\min(m!,n!)}{\max(m!,n!)}}E_\vartheta^{|m-n|/2} \\
&\times \; \mathrm{e}^{-\frac{E_\vartheta}{2}x_+^{(\vartheta)}x_-^{(\vartheta)}}(x_{-sgn(m-n)}^{(\vartheta)})^{|m-n|}\, L_{\min(m,n)}^{|m-n|}\left(E_\vartheta\, x_+^{(\vartheta)}x_-^{(\vartheta)}\right)
\end{aligned} \tag{6.50}$$

with $x_\pm^{(\vartheta)} = t \pm \mathrm{i}\,\mathrm{e}^{-\,\mathrm{i}\,\vartheta}x$ and $L_n^\alpha(z)$ the generalized Laguerre Polynomials.

The proof is given in appendix E. Setting $\vartheta = 0$ this result proves the Euclidean counterpart given in lemma 4.1. Noting that

$$\begin{aligned}
E_\vartheta\, x_+^{(\vartheta)}x_-^{(\vartheta)} &= E\left(\mathrm{e}^{\,\mathrm{i}\,\vartheta}t^2 + \mathrm{e}^{-\,\mathrm{i}\,\vartheta}x^2\right) \\
&= E\left\{\cos(\vartheta)(t^2+x^2) + \mathrm{i}\,\sin(\vartheta)(t^2-x^2)\right\}\,.
\end{aligned} \tag{6.51}$$

we see that similar to the $f_m^{(\gamma_\vartheta)}$ these functions are Schwartz functions only for $|\vartheta| < \pi/2$. In particular they are in $\mathcal{S}_\alpha^\alpha(\mathbb{R}^2)$ for all $\alpha \geq 1/2$. At $\vartheta = \pm\pi/2$ we have a polynomial increase and thus tempered distributions.

The Fock space representation of the harmonic oscillator functions has a counterpart in the complex scaled version, which will be very useful in the explicit determination of the matrix versions of the LS-covariant models. Note that

$$\begin{aligned}
|f_n^{(\gamma_\vartheta)}\rangle\langle f_m^{(\gamma_{-\vartheta})}| &= \hat{\mathbf{V}}(\vartheta)\frac{(\hat{\mathbf{a}}^\dagger)^m}{\sqrt{m!}}|\phi_0\rangle\langle\phi_0|\frac{(\hat{\mathbf{a}})^n}{\sqrt{n!}}\hat{\mathbf{V}}^{-1}(\vartheta) \\
&= \frac{1}{\sqrt{m!n!}}\left(\hat{\mathbf{V}}(\vartheta)\hat{\mathbf{a}}^\dagger\hat{\mathbf{V}}^{-1}(\vartheta)\right)^m |f_0^{(\gamma_\vartheta)}\rangle\langle f_0^{(\gamma_{-\vartheta})}| \left(\hat{\mathbf{V}}(\vartheta)\hat{\mathbf{a}}\hat{\mathbf{V}}^{-1}(\vartheta)\right)^n
\end{aligned} \tag{6.52}$$

with $\hat{\mathbf{a}} = (\hat{\mathbf{q}} + \mathrm{i}\,\hat{\mathbf{p}})/\sqrt{8E}$. We can use relations (6.41) to get

$$\begin{aligned}
\hat{\mathbf{V}}(\vartheta)\hat{\mathbf{a}}^\dagger\hat{\mathbf{V}}^{-1}(\vartheta) &= \mathrm{e}^{\,\mathrm{i}\,\vartheta/2}\hat{\mathbf{q}} - \mathrm{i}\,\mathrm{e}^{-\,\mathrm{i}\,\vartheta/2}\hat{\mathbf{p}}\,, \\
\hat{\mathbf{V}}(\vartheta)\hat{\mathbf{a}}\hat{\mathbf{V}}^{-1}(\vartheta) &= \mathrm{e}^{\,\mathrm{i}\,\vartheta/2}\hat{\mathbf{q}} + \mathrm{i}\,\mathrm{e}^{-\,\mathrm{i}\,\vartheta/2}\hat{\mathbf{p}}\,.
\end{aligned} \tag{6.53}$$

Since $\mathsf{W}\left[\sqrt{2E}t\right] = \hat{\mathbf{q}}$ and $\mathsf{W}\left[\sqrt{2E}x\right] = \hat{\mathbf{p}}$ we find

$$\mathsf{W}\left[\hat{\mathbf{V}}(\vartheta)\hat{\mathbf{a}}^\dagger\hat{\mathbf{V}}^{-1}(\vartheta)\right] = \sqrt{\frac{E_\vartheta}{4}}x_-^{(\vartheta)} \quad , \quad \mathsf{W}\left[\hat{\mathbf{V}}(\vartheta)\hat{\mathbf{a}}\hat{\mathbf{V}}^{-1}(\vartheta)\right] = \sqrt{\frac{E_\vartheta}{4}}x_+^{(\vartheta)}\,. \tag{6.54}$$

where we introduced *generalized light cone coordinates*

$$x_{\pm}^{(\vartheta)} = t \pm \mathrm{i}\, \mathrm{e}^{-\mathrm{i}\vartheta} x \,. \tag{6.55}$$

The corresponding derivatives are given by

$$\partial_{\pm}^{(\vartheta)} = \partial_t \mp \mathrm{i}\, \mathrm{e}^{\mathrm{i}\vartheta} \partial_x \,, \tag{6.56}$$

with $\partial_{\pm}^{(\vartheta)} x_{\pm}^{(\vartheta)} = 2$ and $\partial_{\pm}^{(\vartheta)} x_{\mp}^{(\vartheta)} = 0$. The matrix functions on $\mathbb{R}^2$ can now be obtained via Weyl-Wigner correspondence

$$\begin{aligned} f_{mn}^{(E_\vartheta)} &= \frac{1}{\sqrt{m!n!}} \mathsf{W}\left[|f_m^{(\gamma_\vartheta)}\rangle\langle f_n^{(\gamma_{-\vartheta})}| \right] \\ &= \frac{1}{\sqrt{m!n!}} \left(\sqrt{\frac{E_\vartheta}{4}} x_-^{(\vartheta)} \right)^{\star m} \star \mathsf{W}\left[|f_0^{(\gamma_\vartheta)}\rangle\langle f_0^{(\gamma_{-\vartheta})}| \right] \star \left(\sqrt{\frac{E_\vartheta}{4}} x_+^{(\vartheta)} \right)^{\star n} . \end{aligned} \tag{6.57}$$

Analogue to section 4.2 we define ladder operators through[1]

$$\begin{aligned} \left(\sqrt{\frac{E_\vartheta}{4}} x_-^{(\vartheta)} \right) \star g(\boldsymbol{x}) = a_{(E_\vartheta)}^{+} g(\boldsymbol{x}) \quad &, \quad \left(\sqrt{\frac{E_\vartheta}{4}} x_+^{(\vartheta)} \right) \star g(\boldsymbol{x}) = a_{(E_\vartheta)}^{-} g(\boldsymbol{x}) \,, \\ g(\boldsymbol{x}) \star \left(\sqrt{\frac{E_\vartheta}{4}} x_+^{(\vartheta)} \right) = b_{(E_\vartheta)}^{+} g(\boldsymbol{x}) \quad &, \quad g(\boldsymbol{x}) \star \left(\sqrt{\frac{E_\vartheta}{4}} x_-^{(\vartheta)} \right) = b_{(E_\vartheta)}^{-} g(\boldsymbol{x}) \,. \end{aligned} \tag{6.58}$$

The operators on the rhs can most easily be obtained by expressing the star-product in terms of the generalized light cone coordinates. Inverting the relations (6.56) we get

$$\partial_t = \frac{1}{2}(\partial_+^{(\vartheta)} + \partial_-^{(\vartheta)}) \quad , \quad \partial_x = \frac{\mathrm{e}^{-\mathrm{i}\vartheta}}{2\,\mathrm{i}}(\partial_-^{(\vartheta)} - \partial_+^{(\vartheta)}) \,, \tag{6.59}$$

thus

$$\frac{\mathrm{i}}{E}(\partial_t \partial_x' - \partial_x \partial_t') = \frac{1}{2E_\vartheta}(\partial_+^{(\vartheta)} \partial_-^{\prime(\vartheta)} - \partial_-^{(\vartheta)} \partial_+^{\prime(\vartheta)}) \,. \tag{6.60}$$

The ladder operators are then given by

$$\begin{aligned} a_{(E_\vartheta)}^{\pm} &= \sqrt{\frac{E_\vartheta}{4}} \left(x_{\mp}^{(\vartheta)} \mp \frac{1}{2E_\vartheta} 2\partial_{\pm}^{(\vartheta)} \right) = \frac{1}{2} \left(\sqrt{E_\vartheta} x_{\mp}^{(\vartheta)} \mp \sqrt{\frac{1}{E_\vartheta}} \partial_{\pm}^{(\vartheta)} \right) , \\ b_{(E_\vartheta)}^{\pm} &= \sqrt{\frac{E_\vartheta}{4}} \left(x_{\pm}^{(\vartheta)} \mp \frac{1}{2E_\vartheta} 2\partial_{\mp}^{(\vartheta)} \right) = \frac{1}{2} \left(\sqrt{E_\vartheta} x_{\pm}^{(\vartheta)} \mp \sqrt{\frac{1}{E_\vartheta}} \partial_{\mp}^{(\vartheta)} \right) , \end{aligned} \tag{6.61}$$

and fulfill the relations

$$\left[a_{(E_\vartheta)}^{-}, a_{(E_\vartheta)}^{+} \right] = \frac{1}{2}[\partial_-^{(\vartheta)}, x_-^{(\vartheta)}] = 1 \,, \tag{6.62}$$

$$\left[b_{(E_\vartheta)}^{-}, b_{(E_\vartheta)}^{+} \right] = \frac{1}{2}[\partial_+^{(\vartheta)}, x_+^{(\vartheta)}] = 1 \,, \tag{6.63}$$

whereas all others are zero. We note that the equations derived above are formally identical to those obtained in the Euclidean case in section 4.2, when substituting E_ϑ for E. Of course both coincide for $\vartheta = 0$. The ground state is determined by

$$a_{(E_\vartheta)}^{-} f_{00}^{(E_\vartheta)}(\boldsymbol{x}) = b_{(E_\vartheta)}^{-} f_{00}^{(E_\vartheta)}(\boldsymbol{x}) = 0 \tag{6.64}$$

plus the normalization condition

$$\int \mathrm{d}^2\boldsymbol{x}\, f_{00}^{(E_\vartheta)}(\boldsymbol{x}) = \sqrt{\frac{4\pi}{E}} \tag{6.65}$$

[1] Since $(a_{(E_\vartheta)}^{+})^\dagger \neq a_{(E_\vartheta)}^{-}$ and $(b_{(E_\vartheta)}^{+})^\dagger \neq b_{(E_\vartheta)}^{-}$ they are strictly speaking not ladder operators, but we will nevertheless call them as such.

which has the solution

$$f_{00}^{(E_\vartheta)}(\boldsymbol{x}) = \sqrt{\frac{E}{\pi}}\, \mathrm{e}^{-\frac{E_\vartheta}{2} x_+^{(\vartheta)} x_-^{(\vartheta)}}\,. \tag{6.66}$$

The functions $f_{mn}^{(E_\vartheta)}$ have the ladder operator representation

$$f_{mn}^{(E_\vartheta)}(\boldsymbol{x}) = \frac{(a_{(E_\vartheta)}^+)^m}{\sqrt{m!}} \frac{(b_{(E_\vartheta)}^+)^n}{\sqrt{n!}} f_{00}^{(E_\vartheta)}(\boldsymbol{x})\,. \tag{6.67}$$

It immediately follows that

$$\begin{aligned} a_{(E_\vartheta)}^- f_{mn}^{(E_\vartheta)}(\boldsymbol{x}) &= \sqrt{m} f_{m-1,n}^{(E_\vartheta)}(\boldsymbol{x}) \quad , \quad a_{(E_\vartheta)}^+ f_{mn}^{(E_\vartheta)}(\boldsymbol{x}) = \sqrt{m+1} f_{m+1,n}^{(E_\vartheta)}(\boldsymbol{x})\,, \\ b_{(E_\vartheta)}^- f_{mn}^{(E_\vartheta)}(\boldsymbol{x}) &= \sqrt{n} f_{m,n-1}^{(E_\vartheta)}(\boldsymbol{x}) \quad , \quad b_{(E_\vartheta)}^+ f_{mn}^{(E_\vartheta)}(\boldsymbol{x}) = \sqrt{n+1} f_{m,n+1}^{(E_\vartheta)}(\boldsymbol{x})\,. \end{aligned} \tag{6.68}$$

We will use these relations to obtain the matrix representation of the models in the next section.

Note that the problem of the right test function space is the same as in the generalized oscillator case. The results of the previous section carry over directly to the Wigner transformed case, using the following result [Teo06]:

Lemma 6.2. *Let $\psi \in \mathcal{S}_\alpha^\alpha(\mathbb{R}^d), \varphi \in \mathcal{S}_\alpha^\alpha(\mathbb{R}^d)'$. Then $\psi \in \mathcal{S}_\alpha^\alpha(\mathbb{R}^d)$ if and only if $\mathsf{W}\left[|\psi\rangle\langle\varphi|\right] \in \mathcal{S}_\alpha^\alpha(\mathbb{R}^{2d})$.*

Following this lemma, we can relate the subspaces of Gel'fand-Shilov spaces $\mathcal{S}_\alpha^\alpha(\mathbb{R})$ found in the previous section to subspaces of $\mathcal{S}_\alpha^\alpha(\mathbb{R}^2)$ via Wigner transformation.

6.4 Matrix Model Representation of the Regularized LS-covariant Models

Using the Fock space representation of the last section we will now derive the matrix representation of the classical regularized actions (5.68) and (5.69). In the following we denote

$$f_{mn}^\epsilon \quad := \quad f_{mn}^{(2/\theta_{-\vartheta})} \tag{6.69}$$

with $\vartheta = \pi/2 - \epsilon$. In addition we set $\star = \star_\theta$ with $\theta \neq 2/E$ in general, which means that the generalized Landau functions diagonalize the interaction part, but not necessarily the free part of the action. As in section 4.3 we will assume the fields to be such that the expansion in generalized Landau functions are well-defined.

We expand the scalar fields in terms of the generalized Landau basis

$$\begin{aligned} \phi(\boldsymbol{x}) &= \sum_{mn}^\infty f_{mn}^\epsilon(\boldsymbol{x})\, \phi_{mn}^\epsilon \\ \phi(\boldsymbol{x})^* &= \sum_{mn}^\infty f_{mn}^\epsilon(\boldsymbol{x})\, \overline{\phi}_{mn}^\epsilon \end{aligned} \tag{6.70}$$

where the coefficients given by

$$\begin{aligned} \phi_{mn}^\epsilon &= \int \mathrm{d}^2\boldsymbol{x}\, f_{nm}^\epsilon(\boldsymbol{x})\, \phi(\boldsymbol{x}) \\ \overline{\phi}_{mn}^\epsilon &= \int \mathrm{d}^2\boldsymbol{x}\, f_{nm}^\epsilon(\boldsymbol{x})\, \phi(\boldsymbol{x})^*\,. \end{aligned} \tag{6.71}$$

Using the projector property (6.48) we find

$$f_{m_1n_1}^\epsilon \star f_{m_2n_2}^\epsilon \star f_{m_3n_3}^\epsilon \star f_{m_4n_4}^\epsilon = \frac{1}{\sqrt{2\pi\theta}^3} \delta_{n_1m_2}\delta_{n_2m_3}\delta_{n_3m_4} f_{m_1n_4}^\epsilon\,. \tag{6.72}$$

and thus the LSZ interaction

$$\frac{g}{2\pi\theta}\sum_{mnk\ell}\left(\alpha\,\overline{\phi}^{\epsilon}_{mn}\phi^{\epsilon}_{nk}\,\overline{\phi}^{\epsilon}_{k\ell}\phi^{\epsilon}_{\ell m}+\beta\,\overline{\phi}^{\epsilon}_{mn}\overline{\phi}^{\epsilon}_{nk}\,\phi^{\epsilon}_{k\ell}\phi^{\epsilon}_{\ell m}\right) \tag{6.73}$$

and GW interaction

$$\frac{g}{2\pi\theta}\sum_{mnk\ell}\left(\phi^{\epsilon}_{mn}\phi^{\epsilon}_{nk}\,\phi^{\epsilon}_{k\ell}\phi^{\epsilon}_{\ell m}\right)\,. \tag{6.74}$$

The free parts of the actions can be deduced from the following

Lemma 6.3. *The wave operator of the two-dimensional LSZ model in matrix representation is given by*

$$\begin{aligned}G^{(\epsilon,\sigma)}_{mn;k\ell} &= \left(-\,\mathrm{e}^{-\mathrm{i}\epsilon}\mu^2+2\,\mathrm{i}\,\frac{(1+\Omega^2)}{\theta}(m+n+1)\delta_{m\ell}\delta_{nk}+\frac{4\tilde{\Omega}}{\theta}(n-m)\right)\delta_{m\ell}\delta_{nk}\\ &\quad +2\,\mathrm{i}\,\frac{\Omega^2-1}{\theta}\left(\sqrt{nm}\,\delta_{m,\ell+1}\,\delta_{n,k+1}+\sqrt{(n+1)(m+1)}\,\delta_{m,\ell-1}\,\delta_{n,k-1}\right)\end{aligned} \tag{6.75}$$

with frequencies $\Omega=E\theta/2$ *and* $\tilde{\Omega}=(2\sigma-1)\Omega$.

Proof: The wave operator is defined by

$$G^{(\epsilon,\sigma)}_{mn;k\ell}=\int_x f^{\epsilon}_{mn}(\boldsymbol{x})\left(\sigma\,\mathrm{e}^{\,\mathrm{i}\,\epsilon}\,\mathsf{P}^2(\pi/2-\epsilon)+(1-\sigma)\,\mathrm{e}^{\,\mathrm{i}\,\epsilon}\,\tilde{\mathsf{P}}^2(\pi/2-\epsilon)-\mathrm{e}^{-\mathrm{i}\,\epsilon}\mu^2\right)f^{\epsilon}_{k\ell}(\boldsymbol{x})\,. \tag{6.76}$$

One finds

$$\mathsf{P}^2(\vartheta)=\frac{\mathrm{e}^{\,\mathrm{i}\,\vartheta}}{2\theta}\left[(2+E\theta)^2\left(a^+a^-+\frac{1}{2}\right)+(2-E\theta)^2\left(b^+b^-+\frac{1}{2}\right)+\left(\theta^2E^2-4\right)\left(a^+b^++a^-b^-\right)\right] \tag{6.77}$$

and a similar expression for $\tilde{\mathsf{P}}^2(\vartheta)$ with $a^{\pm}$ and $b^{\pm}$ swapped. The verification of these expressions can be done exactly as in the proof of lemma 4.2 by simply substituting $\theta_{-\vartheta}$ for θ and E_ϑ for B. The matrix representation of $\mathsf{P}^2(\vartheta)$ and $\tilde{\mathsf{P}}^2(\vartheta)$ away from the dual point can be obtained from (6.77) with help of (6.68) leading to

$$\begin{aligned}\mathsf{P}^2_{mn;k\ell}(\vartheta) &= \frac{\mathrm{e}^{\,\mathrm{i}\,\vartheta}}{2\theta}\Bigg[(2+E\theta)^2\left(m+\frac{1}{2}\right)\delta_{m\ell}\,\delta_{nk}+(2-E\theta)^2\left(n+\frac{1}{2}\right)\delta_{m\ell}\,\delta_{nk}\\ &\quad +\left(\theta^2E^2-4\right)\left(\sqrt{nm}\,\delta_{m,\ell+1}\,\delta_{n,k+1}+\sqrt{(n+1)(m+1)}\,\delta_{m,\ell-1}\,\delta_{n,k-1}\right)\Bigg]\end{aligned} \tag{6.78}$$

and

$$\begin{aligned}\tilde{\mathsf{P}}^2_{mn;k\ell}(\vartheta) &= \frac{\mathrm{e}^{\,\mathrm{i}\,\vartheta}}{2\theta}\Bigg[(2+E\theta)^2\left(n+\frac{1}{2}\right)\delta_{m\ell}\,\delta_{nk}+(2-E\theta)^2\left(m+\frac{1}{2}\right)\delta_{m\ell}\,\delta_{nk}\\ &\quad +\left(\theta^2E^2-4\right)\left(\sqrt{nm}\,\delta_{m,\ell+1}\,\delta_{n,k+1}+\sqrt{(n+1)(m+1)}\,\delta_{m,\ell-1}\,\delta_{n,k-1}\right)\Bigg]\,.\end{aligned} \tag{6.79}$$

which can be combined to give (6.75). □

The regularized LSZ model in two-dimensional Minkowski spacetime then has the matrix model representation

$$\mathcal{S}^{(\epsilon)}_{\mathsf{LSZ}} = \sum_{mnk\ell}\overline{\phi}^{\epsilon}_{mn}G^{(\epsilon,\sigma)}_{mn;k\ell}\,\phi^{\epsilon}_{\ell k}+\frac{g}{2\pi\theta}\sum_{mnk\ell}\left(\alpha\,\overline{\phi}^{\epsilon}_{mn}\phi^{\epsilon}_{nk}\,\overline{\phi}^{\epsilon}_{k\ell}\phi^{\epsilon}_{\ell m}+\beta\,\overline{\phi}^{\epsilon}_{mn}\overline{\phi}^{\epsilon}_{nk}\phi^{\epsilon}_{k\ell}\,\phi^{\epsilon}_{\ell m}\right)\,. \tag{6.80}$$

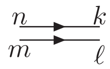
A perturbative expansion of the generating functional in matrix basis is similarly obtained as in the Euclidean case 4.4. The generating functional of the LSZ model reads

$$\begin{aligned}Z[J] &= \lim_{\epsilon\to 0^+}\mathcal{N}\exp\left(-\,\mathrm{i}\,\alpha\,g\sum_{mnk\ell}\frac{\partial^4}{\partial J^{\epsilon}_{m\ell}\partial\bar{J}^{\epsilon}_{\ell k}\partial J^{\epsilon}_{kn}\partial\bar{J}^{\epsilon}_{nm}}\right)\\ &\quad\times\ \exp\left(-\,\mathrm{i}\,\beta\,g\sum_{mnk\ell}\frac{\partial^4}{\partial J^{\epsilon}_{m\ell}\partial J^{\epsilon}_{\ell k}\partial\bar{J}^{\epsilon}_{kn}\partial\bar{J}^{\epsilon}_{nm}}\right)\exp\left(\frac{\mathrm{i}}{2}\sum_{mnk\ell}\bar{J}^{\epsilon}_{mn}\Delta^{(\epsilon,\sigma)}_{mn;k\ell}J^{\epsilon}_{k\ell}\right)\,,\end{aligned} \tag{6.81}$$

with J^{ϵ}_{mn} and $\bar{J}^{\epsilon}_{mn}$ the sources in matrix basis and the propagator $\Delta^{(\epsilon,\sigma)}_{mn;k\ell}$ defined as the inverse of $G^{(\epsilon,\sigma)}_{mn;k\ell}$:

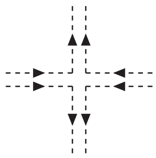
$$\sum_{k\ell} G^{(\epsilon,\sigma)}_{mn;k\ell}\Delta^{(\epsilon,\sigma)}_{\ell k;sr} = \sum_{k\ell} \Delta^{(\epsilon,\sigma)}_{nm;\ell k} G^{(\epsilon,\sigma)}_{k\ell;rs} = \delta_{mr}\delta_{ns}\,. \tag{6.82}$$

The modified Feynman rules are presented in the double line formalism and are exactly as in the Euclidean case. The double lines are oriented pointing from ϕ^* to ϕ :

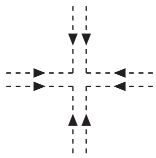


$$= \Delta^{(\epsilon,\sigma)}_{nm;\ell k} \quad .$$

The two interaction terms $\phi^* \star \phi \star \phi^* \star \phi$ and $\phi^* \star \phi^* \star \phi \star \phi$ are represented by different diagrams



$$\sim \phi^* \star \phi \star \phi^* \star \phi$$



$$\sim \phi^* \star \phi^* \star \phi \star \phi$$

having vertices $-\,\mathrm{i}\, g\, \delta_{mp}\delta_{nq}\delta_{kr}\delta_{\ell s}$ times α or β, respectively.

The GW model can be treated identically. One can immediately follow from lemma (6.3) by setting $\sigma = 1/2$:

Lemma 6.4. *The regularized Grosse-Wulkenhaar wave operator in two dimensions has the matrix representation given by*

$$\begin{aligned} G^{(\epsilon)}_{mn;k\ell} \quad = \quad & \left(-\,\mathrm{e}^{-\,\mathrm{i}\,\epsilon}\mu^2 + 2\,\mathrm{i}\,\frac{\Omega^2+1}{\theta}(m+n+1)\right)\delta_{m\ell}\,\delta_{nk} \\ & +2\,\mathrm{i}\,\frac{\Omega^2-1}{\theta}\left(\sqrt{nm}\delta_{m,\ell+1}\,\delta_{n,k+1} + \sqrt{(n+1)(m+1)}\delta_{m,\ell-1}\delta_{n,k-1}\right) \end{aligned} \tag{6.83}$$

with frequency $\Omega = E\theta/2$.

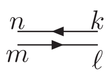
The Minkowskian GW action then reads

$$\mathcal{S}^{(\epsilon)}_{GW} = \sum_{mn;k\ell}\left(\frac{1}{2}\phi^{\epsilon}_{mn}G^{(\epsilon)}_{mn;k\ell}\phi^{\epsilon}_{k\ell} + \frac{g}{2\pi\theta}\phi^{\epsilon}_{mn}\phi^{\epsilon}_{nk}\phi^{\epsilon}_{k\ell}\phi^{\epsilon}_{\ell m}\right)\,. \tag{6.84}$$

The generating functional is given by

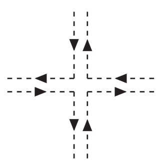
$$Z[J] = \lim_{\epsilon\to 0}\mathcal{N}\,\exp\left(-\,\mathrm{i}\,g\sum_{mnk\ell}\frac{\partial^4}{\partial J^{\epsilon}_{m\ell}\partial J^{\epsilon}_{\ell k}\partial J^{\epsilon}_{kn}\partial J^{\epsilon}_{nm}}\right)\exp\left(\frac{\mathrm{i}}{2}\sum_{mnk\ell}J^{\epsilon}_{mn}\Delta^{(\epsilon)}_{mn;k\ell}J^{\epsilon}_{k\ell}\right) \tag{6.85}$$

with the propagator $\Delta^{(\epsilon)}_{mn;k\ell}$ being the inverse of $G^{(\epsilon)}_{mn;k\ell}$ and being represented by the unoriented double line



$$= \Delta^{(\epsilon)}_{nm;\ell k}\,.$$

The vertex of the $\phi^{\star 4}$ interaction is given by the graph



$$= -\,\mathrm{i}\,g\,\delta_{mp}\delta_{nq}\delta_{kr}\delta_{\ell s}\,.$$

Since the vertex is oriented there will be as many diagrams as in the LSZ action with both parameters α and β turned on.

6.5 Generalization to Higher Dimensions

The generalization to higher dimensions can be obtained similarly as in section 4.6. By definition, the $D = 2n$-dimensional operators $\mathsf{K}^2(\vartheta)$ and $\tilde{\mathsf{K}}^2(\vartheta)$ are given by

$$\begin{aligned} \mathsf{K}^2(\vartheta) &= \mathsf{P}^2(\vartheta) + \mathrm{e}^{2\,\mathrm{i}\,\vartheta} \sum_{k=2}^{n} (\mathsf{P}_i^2)_k \\ \tilde{\mathsf{K}}^2(\vartheta) &= \tilde{\mathsf{P}}^2(\vartheta) + \mathrm{e}^{2\,\mathrm{i}\,\vartheta} \sum_{k=2}^{n} (\tilde{\mathsf{P}}_i^2)_k \,, \end{aligned} \tag{6.86}$$

according to equations (6.2) and (6.3). We found that the spectra of both operators are given by

$$\{4E\,\mathrm{e}^{\,\mathrm{i}\,\vartheta}(\ell_1 + 1/2) + \sum_{k=2}^{n} 4B_k\,\mathrm{e}^{2\,\mathrm{i}\,\vartheta}(\ell_k + 1/2)\,,\, \ell_1, \dots, \ell_n \in \mathbb{N}\}\,, \tag{6.87}$$

where the eigenfunctions are products of generalized Landau functions from section 6.3 $f^{(E_\vartheta)}_{m_1 n_1}$ and ordinary Landau functions from section 4.2 $f^{(B_k)}_{m_k n_k}$:

$$f^{(\boldsymbol{F}_\vartheta)}_{\boldsymbol{mn}}(\boldsymbol{x}) := f^{(E_\vartheta)}_{m_1 n_1}(\boldsymbol{x}_1) f^{(B_2)}_{m_2 n_2}(\boldsymbol{x}_2) \cdots f^{(B_n)}_{m_n n_n}(\boldsymbol{x}_n) \tag{6.88}$$

with $\boldsymbol{x}_k = (x^{2k-2}, x^{2k-1}) \in \mathbb{R}^2$, $\boldsymbol{x} = (\boldsymbol{x}^\mu) \in \mathbb{R}^D$, $\boldsymbol{m} = (m_k), \boldsymbol{n} = (n_k) \in \mathbb{N}^n$ and $\boldsymbol{F}_\vartheta = (E_\vartheta, B_2, \dots, B_n) \in \mathbb{C}_+ \times \mathbb{R}^n_+$, where $\mathbb{C}_+$ denotes the complex numbers with positive real part. The deformation matrix Θ is assumed to be in its canonical form

$$(\Theta^{\mu\nu}) = \begin{pmatrix} \begin{matrix} 0 & \theta_1 \\ -\theta_1 & 0 \end{matrix} & & 0 \\ & \ddots & \\ 0 & & \begin{matrix} 0 & \theta_n \\ -\theta_n & 0 \end{matrix} \end{pmatrix} \tag{6.89}$$

with $\theta_i \in \mathbb{R}$. The star product of two such multi-dimensional, generalized Landau functions with respect to (6.89) decouples into products of Landau functions depending on $\boldsymbol{x}_k$ for $k = 1, \dots, n$. If in addition $E = 2/\theta$ and $B_k = 2/\theta_k$ for all k, then

$$\left(f^{(\boldsymbol{F}_\vartheta)}_{\boldsymbol{mn}} \star_\Theta f^{(\boldsymbol{F}_\vartheta)}_{\boldsymbol{m}'\boldsymbol{n}'} \right)(\boldsymbol{x}) = \delta_{\boldsymbol{nm}'}\, f^{(\boldsymbol{F}_\vartheta)}_{\boldsymbol{mn}'}(\boldsymbol{x}) \tag{6.90}$$

with $\delta_{\boldsymbol{m}'\boldsymbol{n}} = \prod_{k=1}^n \delta_{m'_k n_k}$.

The generalization of the matrix model representation is straightforward. To confirm with our previous notation we set $\vartheta = \pi/2 - \epsilon > 0$ and use the notation

$$f^\epsilon_{\boldsymbol{mn}}(\boldsymbol{x}) = \prod_{k=1}^{n} f^{(2/(\theta_k) - \vartheta)}_{m_k n_k}(\boldsymbol{x})\,. \tag{6.91}$$

The $f^\epsilon_{\boldsymbol{mn}}$ are arranged such as to simplify the interaction part but not necessary the free part of the action. The scalar fields living on $\mathbb{R}^D$ are expanded in the generalized Landau basis

$$\begin{aligned} \phi(\boldsymbol{x}) &= \sum_{\boldsymbol{m},\boldsymbol{n} \in \mathbb{N}^n}^{\infty} f^\epsilon_{\boldsymbol{mn}}(\boldsymbol{x})\, \phi^\epsilon_{\boldsymbol{mn}} \\ \phi(\boldsymbol{x})^* &= \sum_{\boldsymbol{m},\boldsymbol{n} \in \mathbb{N}^n}^{\infty} f^\epsilon_{\boldsymbol{mn}}(\boldsymbol{x})\, \overline{\phi}^\epsilon_{\boldsymbol{mn}} \end{aligned} \tag{6.92}$$

where the coefficients are given by

$$\begin{aligned} \phi^\epsilon_{\boldsymbol{mn}} &= \int \mathrm{d}^D\boldsymbol{x}\, f^\epsilon_{\boldsymbol{nm}}(\boldsymbol{x})\, \phi(\boldsymbol{x}) \\ \overline{\phi}^\epsilon_{\boldsymbol{mn}} &= \int \mathrm{d}^D\boldsymbol{x}\, f^\epsilon_{\boldsymbol{nm}}(\boldsymbol{x})\, \phi(\boldsymbol{x})^*\,. \end{aligned} \tag{6.93}$$

The matrix representation of the $D = 2n$-dimensional LSZ model away from the self-dual point can be obtained by comparing the operators (6.86) with its two dimensional constituents and their matrix representations given by the equations (4.50), (4.51), (6.78) and (6.79). The matrix LSZ operator is thus the sum of the two-dimensional Minkowskian case given by (6.75) plus $n-1$ copies of the *massless* Euclidean operator given by (4.45) times $\mathrm{e}^{-\mathrm{i}\,\epsilon}$, where we set again $\vartheta = \pi/2 - \epsilon$. Noting that the massless LSZ operators in Euclidean and Minkowskian space differ only by a factor "i", we can write

$$G^{(\epsilon,\sigma)}_{\boldsymbol{mn};\boldsymbol{k\ell}} = \mathrm{i}\,\mathcal{G}^{(\sigma)}_{m_1n_1;k_1\ell_1} - \mathrm{e}^{-\mathrm{i}\,\epsilon}\sum_{i=2}^{n}\mathcal{G}^{(\sigma)}_{m_in_i;k_i\ell_i} - \mathrm{e}^{-\mathrm{i}\,\epsilon}\mu^2 \tag{6.94}$$

with $\boldsymbol{m} = (m_k), \boldsymbol{n} = (n_k), \boldsymbol{k} = (k_k), \boldsymbol{\ell} = (\ell_k) \in \mathbb{N}^n$ and $\mathcal{G}_{mn,k\ell}$ the two dimensional, massless, Euclidean LSZ matrix wave operators

$$\begin{aligned}\mathcal{G}^{(\sigma)}_{mn;k\ell} \quad &= \quad \left(2\frac{\Omega^2+1}{\theta}(m+n+1) + \frac{4\tilde{\Omega}}{\theta}(n-m)\right)\delta_{m\ell}\,\delta_{n,k} \\ &\qquad + 2\frac{\Omega^2-1}{\theta}\left(\sqrt{nm}\delta_{m,\ell+1}\,\delta_{n,k+1} + \sqrt{(n+1)(m+1)}\delta_{m,\ell-1}\delta_{n,k-1}\right)\end{aligned} \tag{6.95}$$

with $\Omega = E\theta/2 = B_i\theta/2$ and $\tilde{\Omega} = (2\sigma-1)\Omega$. The $2n$-dimensional, regularized LSZ action is then given in the usual form

$$\begin{aligned}\mathcal{S}_{\mathsf{LSZ}} \quad &= \quad \sum_{\boldsymbol{m},\boldsymbol{n},\boldsymbol{k},\boldsymbol{\ell}\,\in\,\mathbb{N}^n}\overline{\phi}^{\epsilon}_{\boldsymbol{mn}}G^{(\epsilon,\sigma)}_{\boldsymbol{mn};\boldsymbol{k\ell}}\,\phi^{\epsilon}_{\boldsymbol{\ell k}} \\ &\qquad + \frac{g}{2\pi\theta}\sum_{\boldsymbol{m},\boldsymbol{n},\boldsymbol{k},\boldsymbol{\ell}\,\in\,\mathbb{N}^n}\left(\alpha\,\overline{\phi}^{\epsilon}_{\boldsymbol{mn}}\phi^{\epsilon}_{\boldsymbol{nk}}\,\overline{\phi}^{\epsilon}_{\boldsymbol{k\ell}}\phi^{\epsilon}_{\boldsymbol{\ell m}} + \beta\,\overline{\phi}^{\epsilon}_{\boldsymbol{mn}}\overline{\phi}^{\epsilon}_{\boldsymbol{nk}}\phi^{\epsilon}_{\boldsymbol{k\ell}}\,\phi^{\epsilon}_{\boldsymbol{\ell m}}\right)\,.\end{aligned} \tag{6.96}$$

Every other result of this chapter can now formally be generalized to higher dimensions by substituting multi-indices $\boldsymbol{m}, \boldsymbol{n}, \ldots \in \mathbb{N}^n$ for usual one-dimensional indices $m, n, \ldots \in \mathbb{N}$.

7 Aspects of the LS-Covariant Theories

In this chapter we will treat several questions concerning the LS-covariant models in Minkowski spacetime, like the determination of the causal propagator, LS-covariance at quantum level and unitarity. Problematic for the propagator and the unitarity issue turns out to be the lack of translation invariance, which manifests itself in an instability of the vacuum with respect to pair production. We review how the ordinary procedures, one is used to, have to be altered to take care of these features. In addition, the two different possibilities to treat these models, the continuous and the matrix basis, will be compared. We will first comment on the corresponding propagators, which one obtains by removing the ϑ-regularization. Afterwards we discuss their applicability to Feynman diagrams in the case $\Omega = 1$. The question of how to implement LS-duality at quantum level is given in section 7.3. The unitarity of the LS-covariant models will be discussed in section 7.4.

7.1 Causal Propagator

It is a feature of all frequently considered physical theories on Minkowski spacetime that there is more than one propagator, that means a function (or distribution) Δ which solves the equation $\mathsf{D}_x \Delta(\boldsymbol{x}, \boldsymbol{x}') = \delta(\boldsymbol{x} - \boldsymbol{x}')$ with D_x being the wave operator of the theory. Any two of these differ by a solution of the equation of motion. It is therefore necessary to impose further conditions as to make the solution of this problem unique. This may be done by imposing boundary conditions, postulating a spectral representation or extending the wave operator as to make the equation unique. We are mainly concerned with the question which propagators show up in the generating functionals (5.74) and (5.75). The free generating functional $Z_0[J]$ is defined as the vacuum-to-vacuum amplitude

$$Z_0[J] = \langle \Omega, out | \Omega, in \rangle [J]\,, \tag{7.1}$$

where $|\Omega, in\rangle$ and $\langle \Omega, out|$ are the vacua at time instances t_{in} and t_{out} of the quantum theory defined by $\mathcal{S}_0[\varphi]$ in presence of the source J. Using Schwinger's action principle, one can show that *causality* implies

$$\left.\frac{\delta^2 \log Z_0[J]}{\delta J(\boldsymbol{x})\, \delta J(\boldsymbol{y})}\right|_{J=0} = \frac{\langle 0, out | T(\hat{\phi}(\boldsymbol{x}) \hat{\phi}^\dagger(\boldsymbol{y})) | 0, in\rangle}{\langle 0, out | 0, in\rangle}\,, \tag{7.2}$$

where $\hat{\phi}$ is the field operator and $|0, in\rangle$ and $\langle 0, out|$ the in- and out- vacua for $J = 0$, which in the presence of further interactions are supposed to be in the interaction picture with respect to $\mathcal{S}_0[\varphi]$. Note that for theories which allow spontaneous pair production, which is the case for the LS-covariant models we are considering, the in- and out- vacua are in general not dual to each other, thus $|\langle 0, out | 0, in\rangle| < 1$ which has to be taken into account. This is evident, since $\langle 0, out | 0, in\rangle$ measures the vacuum persistence and is equal to 1 only if no spontaneous pair production occurs. The rhs is known as *causal* propagator and will be denoted as $\mathrm{i}\,\Delta_c$, where the imaginary unit has been factored out for convenience. Quite generally, for a Klein-Gordon field, which may be free or moving in an external background which preserves vacuum stability, the expression (7.2) may be evaluated as

$$\begin{aligned} \mathrm{i}\,\Delta_c(\boldsymbol{x}, \boldsymbol{x}') &= \theta(x^0 - x'^0) \sum_n \phi_n^{(+)}(\boldsymbol{x}) \overline{\phi}_n^{(+)}(\boldsymbol{x}') \\ &+ \theta(x'^0 - x^0) \sum_n \phi_n^{(-)}(\boldsymbol{x}) \overline{\phi}_n^{(-)}(\boldsymbol{x}') \end{aligned} \tag{7.3}$$

with $(\phi_n^{(\pm)})$ being a complete set of *solutions* of the equation of motion with positive and negative frequency, respectively, and n being an index comprising the quantum numbers. One can check that (7.2) propagates particles (positive frequency solutions) forward in time and anti-particles (negative frequency solutions) backward. This is the imprint of causality and lends the causal propagator its name.

The situation gets more complicated if the background field spoils vacuum persistence. Crucial for the canonical quantization scheme and for equation (7.3) to be applicable is the existence of a complete set of solutions, which allows for a distinction between positive or negative frequencies through all times. However, such a set of solutions only exists if we are dealing with a "stationary spacetime", which says that the spacetime allows for a global timelike Killing vector field [DeW75]. In our case, there does not exist such a vector field due to the lack of time translation symmetry. The methods to be used have been developed in [Git77, FG81]. Since the asymptotic Hilbert spaces in the remote past and future, provided they exist, are different, we have two sets of solutions, denoted as $({\phi_n}^{(\pm)})_n$ and $(\phi_{n(\pm)})_n$ and being the equivalent to positive/negative frequency solutions above in the infinite future and past, respectively. The generalization of the sum over solutions (7.3) then reads

$$\begin{aligned} \mathrm{i}\,\Delta_c(\boldsymbol{x},\boldsymbol{x}') &= \theta(x^0-x'^0)\sum_{m,n}{\phi_m}^{(+)}(\boldsymbol{x})\,\omega(m^+|n^+)\,\overline{\phi}_{n(+)}(\boldsymbol{x}') \\ &+ \theta(x'^0-x^0)\sum_{m,n}\phi_{n(-)}(\boldsymbol{x})\,\omega(m^-|n^-)\,{\overline{\phi}_m}^{(-)}(\boldsymbol{x}')\,, \end{aligned} \tag{7.4}$$

with $\omega(m^\pm|n^\pm)$ being the relative probability for a particle/anti-particle to be scattered by vacuum (see also section 7.4). For a theory with a stable vacuum this is just δ_{mn} and in addition ${\phi_n}^{(\pm)} = \phi_{n(\pm)}$. This procedure determines the propagator uniquely and is equal to the definition (7.2), but might at times be quite complicated to perform, for which it is desirable to have another method at hand.

Such an equivalent method, which will proves profitable for us, is the *eigenvalue representation*. Let $\varphi_n(\boldsymbol{x})$ be an orthonormal and complete set of eigenfunctions of the wave operator D_x with eigenvalues λ_n, i.e.

$$\mathsf{D}_x\varphi_n(\boldsymbol{x}) = \lambda_n\varphi_n(\boldsymbol{x}) \tag{7.5}$$

with

$$\sum_n \overline{\varphi}_n(\boldsymbol{x})\,\varphi_n(\boldsymbol{x}') = \delta(\boldsymbol{x}-\boldsymbol{x}') \quad \text{and} \quad \int_x \overline{\varphi}_n(\boldsymbol{x})\,\varphi_m(\boldsymbol{x}) = \delta_{nm}\,. \tag{7.6}$$

Note that, contrary to the $\phi_n^{(\pm)}$ above, these eigenfunctions may not solve the equations of motion. Decomposing the propagator into these eigenfunctions gives formally

$$\Delta(\boldsymbol{x},\boldsymbol{x}') = \sum_n \overline{\varphi}_n(\boldsymbol{x})\lambda_n^{-1}\varphi_n(\boldsymbol{x}')\,, \tag{7.7}$$

however singularities at $\lambda_n = 0$ for any n pose problems to this definition, which reflects the fact of having more than one propagator for a single theory. Usually one modifies the denominator by a small imaginary part $\lambda_n \to \lambda_n + \mathrm{i}\,\epsilon f(n)$ with small $\epsilon > 0$ and $f(n)$ some function such that

$$\lambda_n + \mathrm{i}\,\epsilon f(n) \neq 0 \quad , \quad \forall n\,. \tag{7.8}$$

A propagator for D_x is finally obtained by taking the limit $\epsilon \to 0$. Equivalently one can regularize the operator $\mathsf{D}_x \to \mathsf{D}_x^{(\epsilon)}$ with $\lim_{\epsilon\to 0}\mathsf{D}_x^{(\epsilon)} = \mathsf{D}_x$ and solve the equation

$$\mathsf{D}_x^{(\epsilon)}\Delta^{(\epsilon)}(\boldsymbol{x},\boldsymbol{x}') = \delta(\boldsymbol{x}-\boldsymbol{x}')\,, \tag{7.9}$$

where $\lim_{\epsilon\to 0^+}\Delta^{(\epsilon)}(\boldsymbol{x},\boldsymbol{x}')$ is a propagator of the original operator D_x. Hence any well-defined operator which is continuously connected to the original operator and has no zero eigenvalue gives rise to a propagator for D_x. However, apart from the absence of zero eigenvalues of $\mathsf{D}_x^{(\epsilon)}$, or equivalently condition (7.8), the regularization is arbitrary, and different regularizations may lead to different propagators. For example in the free Klein-Gordon case $f(\boldsymbol{k}) = const. > 0$ leads to the Feynman propagator, while $f(\boldsymbol{k}) = 2k^0$ yields the retarded propagator. In general one cannot be sure whether one got the causal propagator unless one compares it to the result obtained from (7.2). This is the obvious problem of the eigenvalue method, and it is still not solved for the general case of any propagator and any external field.

For the LSZ model the two different regularized operators are

$$\begin{aligned} \mathsf{D}^{(\epsilon)}_{x,disc} &= \sigma(\mathsf{K}_\mu^2-\mu^2)_\epsilon + (1-\sigma)(\tilde{\mathsf{K}}_\mu^2-\mu^2)_\epsilon & (7.10)\\ \mathsf{D}^{(\epsilon)}_{x,cont} &= \sigma\mathsf{K}_\mu^2 + (1-\sigma)\tilde{\mathsf{K}}_\mu^2 - \mu^2 + \mathrm{i}\,\epsilon\,, & (7.11) \end{aligned}$$

introduced in section 5.4. The question to which propagator they lead in the limit $\epsilon \to 0$ has been answered for the $\mathrm{i}\,\epsilon$-prescription for several related models. For the KG field moving in crossed or parallel uniform electric and magnetic fields, or in an electric field with an additional plane wave, this method gives the causal propagator [Rit70, Rit78, BFS85]. Since an additional uniform, constant magnetic background should not change the pole structure of the propagator, we do not doubt that the $\mathrm{i}\,\epsilon$-prescription will also give the causal propagator in the case of a pure electric field. In the next section we will confirm that the ϑ-regularization (7.10) gives the same propagator.

7.1.1 Propagator from Matrix Regularization

The equivalence of the propagators in the different representations have to be checked by hand, which is easily done in the free case. For generic electromagnetic backgrounds this is still an open question, and in order to make a comparison we have to restrict to cases where the propagators are already known. Using the "sum over solutions method" (7.4), the causal propagator for a scalar field in four dimensions with a constant, uniform electric field along one space direction has been calculated in [FGS91] (equation (6.2.40)):

$$\begin{aligned} \Delta_c(\boldsymbol{x},\boldsymbol{x}') = \frac{eE}{16\pi^2}\,\mathrm{e}^{\,\mathrm{i}\,\frac{e}{2}\boldsymbol{x}_\parallel\cdot\boldsymbol{E}\cdot\boldsymbol{x}'_\parallel} &\int_0^\infty \frac{\mathrm{d}s}{s}\frac{1}{\sinh(sE)} \\ \times\quad \exp&\left\{-\,\mathrm{i}\,s\mu^2 - \frac{\mathrm{i}}{2}eE(\boldsymbol{x}_\parallel-\boldsymbol{x}'_\parallel)^2\coth(seE) + \mathrm{i}\,\frac{(\boldsymbol{x}_\perp-\boldsymbol{x}'_\perp)^2}{4s}\right\}. \end{aligned} \tag{7.12}$$

Here we defined $\boldsymbol{x} = (\boldsymbol{x}_\parallel, \boldsymbol{x}_\perp) \in \mathbb{R}^4$ with $\boldsymbol{x}_\perp$ denoting the two space components perpendicular to the electric field and

$$\boldsymbol{x}_\parallel\cdot\boldsymbol{E}\cdot\boldsymbol{x}'_\parallel := E(x_\parallel)^\mu\epsilon_{\mu\nu}(x'_\parallel)^\nu\ , \tag{7.13}$$

where $\epsilon_{\mu\nu}$ is the two-dimensional Levi-Civita-tensor with $\epsilon_{01}=1$ and $E>0$ the electric field strength. Below we will start with this four-dimensional wave operator, where the electric part is regularized as in (7.10), and calculate its (unique) propagator. For $\epsilon\to 0$ we find coincidence with (7.12) confirming that this is the causal propagator. This result can easily be carried over to the two-dimensional case confirming that the ϑ-regularization leads to causality for the LSZ model at $\sigma=1$. We conjecture that this also holds for $\sigma\neq 1$. The calculations done here using the matrix basis are comparable simple, such that the matrix basis can be seen as a powerful computational tool.

We define the map $(\cdot,\cdot)_\vartheta : \mathbb{R}^2\times\mathbb{R}^2 \to \mathbb{C}$ for $\vartheta\in[-\pi/2,\pi/2]$ by

$$(\boldsymbol{x},\boldsymbol{x}')_\vartheta = \cos(\vartheta)\,(\boldsymbol{x},\boldsymbol{x}')_E + \mathrm{i}\,\sin(\vartheta)\,(\boldsymbol{x},\boldsymbol{x}')_M\ , \tag{7.14}$$

where $(\cdot,\cdot)_M$ is the two dimensional Minkowskian and $(\cdot,\cdot)_E$ the two dimensional Euclidean scalar product. In addition we define the map $\|\cdot\| : \mathbb{R}^2 \to \mathbb{C}$ by

$$\begin{aligned} \|\boldsymbol{x}\|^2_\vartheta &= (\boldsymbol{x},\boldsymbol{x})_\vartheta \\ &= \cos(\vartheta)\|\boldsymbol{x}\|_E + \mathrm{i}\,\sin(\vartheta)\|\boldsymbol{x}\|_M \end{aligned} \tag{7.15}$$

with $\|\cdot\|_E$ the two dimensional Euclidean and $\|\cdot\|_M$ the two dimensional Minkowskian norm. For arbitrary two-dimensional vectors $\boldsymbol{x},\boldsymbol{x}'\in\mathbb{R}^2$ we denote as above

$$\boldsymbol{x}\cdot\boldsymbol{E}\cdot\boldsymbol{x}' = Ex^\mu\epsilon_{\mu\nu}x'^\nu\ . \tag{7.16}$$

We need the following lemma

Lemma 7.1. *Let $\boldsymbol{x} \in \mathbb{R}^2$ and $a \in \mathbb{C} - \{0\}$. The following identity holds*

$$\sum_{n=0}^{\infty} f_{mn}^{(E_\vartheta)}(\boldsymbol{x})\, f_{nm}^{(E_\vartheta)}(\boldsymbol{x}')a^n = \frac{E}{\pi}\exp\left\{-\frac{E}{2}\|\boldsymbol{x}-\boldsymbol{x}'\|_\vartheta^2 + (a-1)E(\boldsymbol{x},\boldsymbol{x}')_\vartheta - a\,\mathrm{i}\,\boldsymbol{x}\cdot\boldsymbol{E}\cdot\boldsymbol{x}'\right\}$$
$$\times\; L_m\left(E\|\boldsymbol{x}-\boldsymbol{x}'\|_\vartheta^2 - a(1-a^{-1})^2E(\boldsymbol{x},\boldsymbol{x}')_\vartheta + (a-a^{-1})\,\mathrm{i}\,\boldsymbol{x}\cdot\boldsymbol{E}\cdot\boldsymbol{x}'\right). \tag{7.17}$$

Proof: is given in appendix G.

An immediate corollary is

Corollary 7.2. *The following relations hold*

$$\sum_{n=0}^{\infty} f_{mn}^{(E_\vartheta)}(\boldsymbol{x})\, f_{nm}^{(E_\vartheta)}(\boldsymbol{x})a^n = \frac{E}{\pi}\,\mathrm{e}^{(a-1)E(\boldsymbol{x},\boldsymbol{x}')_\vartheta}a^m\, L_m\left(-E\frac{(a-1)^2}{a}\|\boldsymbol{x}\|_\vartheta^2\right) \tag{7.18}$$

$$\sum_{n=0}^{\infty} f_{mn}^{(E_\vartheta)}(\boldsymbol{x})\, f_{nm}^{(E_\vartheta)}(\boldsymbol{x}') = \frac{E}{\pi}\exp\left\{-\frac{E}{2}\|\boldsymbol{x}-\boldsymbol{x}'\|_\vartheta^2 - \mathrm{i}\,\boldsymbol{x}\cdot\boldsymbol{E}\cdot\boldsymbol{x}'\right\}L_m\left(E\|\boldsymbol{x}-\boldsymbol{x}'\|_\vartheta^2\right) \tag{7.19}$$

$$\sum_{n=0}^{\infty} f_{mn}^{(E_\vartheta)}(\boldsymbol{x})\, f_{nm}^{(E_\vartheta)}(\boldsymbol{x}) = \frac{E}{\pi}\,. \tag{7.20}$$

Now we determine the propagator of the Klein-Gordon field in four dimensions exposed to a constant electric field, where the wave operator parallel to the electric field is given by the two-dimensional, regularized operator $\left(\mathsf{P}_\mu^2-\mu^2\right)_\epsilon$. The coordinate vector is again written as $\boldsymbol{x} = (\boldsymbol{x}_\parallel, \boldsymbol{x}_\perp)$ with $\boldsymbol{x}_\perp$ being the components perpendicular to the electric field, and analogously for the momenta $\boldsymbol{p} = (\boldsymbol{p}_\parallel, \boldsymbol{p}_\perp)$ and derivatives $\partial_\mu = (\partial_\parallel, \partial_\perp)$.

Theorem 7.3. *The propagator of the regularized wave operator $\mathsf{D}_x^{(\epsilon)} = \left(\mathsf{P}_\mu^2-\mu^2\right)_\epsilon + (\mathrm{i}\,\partial_\perp)^2$ coincides in the limit $\epsilon \to 0$ with the causal propagator (7.12).*

Proof: The inverse of $\mathsf{D}_x^{(\epsilon)}$ is given by

$$\Delta^{(\epsilon)}(\boldsymbol{x},\boldsymbol{x}') = \langle\boldsymbol{x}|\frac{1}{(\mathsf{P}_\mu^2-\mu^2)_\epsilon + (\mathrm{i}\,\partial_\perp)^2}|\boldsymbol{x}'\rangle\,, \tag{7.21}$$

where $(\mathsf{P}_\mu^2-\mu^2)_\epsilon + (\mathrm{i}\,\partial_\perp)^2 = \mathrm{e}^{\,\mathrm{i}\,\epsilon}\mathsf{P}^2(\pi/2-\epsilon) - \mathrm{e}^{-\mathrm{i}\,\epsilon}\mu^2 + (\mathrm{i}\,\partial_\perp)^2$ with $\epsilon > 0$ fulfills the eigenvalue equation

$$\left[\left(\mathsf{P}_\mu^2-\mu^2\right)_\epsilon + (\mathrm{i}\,\partial_\perp)^2\right] f_{mn}^{(E_\vartheta)}(\boldsymbol{x}_\parallel)\,\mathrm{e}^{-\mathrm{i}\,\boldsymbol{p}_\perp\cdot\boldsymbol{x}_\perp}$$
$$= \left[\mathrm{i}\,4E\left(m+\frac{1}{2}\right) + \boldsymbol{p}_\perp - \mathrm{e}^{-\mathrm{i}\,\epsilon}\mu^2\right] f_{mn}^{(E_\vartheta)}(\boldsymbol{x}_\parallel)\,\mathrm{e}^{-\mathrm{i}\,\boldsymbol{p}_\perp\cdot\boldsymbol{x}_\perp} \tag{7.22}$$

with $\vartheta = \pi/2-\epsilon$. We simply write μ^2 for $\mathrm{e}^{-\mathrm{i}\,\epsilon}\mu^2$, keeping in mind that μ^2 is slightly imaginary. Using the identity

$$\frac{1}{a} = -\mathrm{i}\int_0^\infty \mathrm{d}s\,\mathrm{e}^{\,\mathrm{i}\,sa} \quad , \mathfrak{Im}(a) > 0\,, \tag{7.23}$$

we obtain

$$\langle\boldsymbol{x}|\left(\frac{1}{\mathsf{P}_\mu^2-\mu^2}\right)_\epsilon|\boldsymbol{x}'\rangle = -\mathrm{i}\int_0^\infty \mathrm{d}s\int\frac{\mathrm{d}^2\boldsymbol{p}_\perp}{(2\pi)^2}\sum_{m,n=0}^{\infty} f_{mn}^{(E_\vartheta)}(\boldsymbol{x}_\parallel)f_{nm}^{(E_\vartheta)}(\boldsymbol{x}'_\parallel)$$
$$\times\;\mathrm{e}^{-\mathrm{i}\,s\mu^2}\,\mathrm{e}^{-s4E(m+\frac{1}{2})}\,\mathrm{e}^{\,\mathrm{i}\,s\boldsymbol{p}_\perp^2 - \mathrm{i}\,(\boldsymbol{x}_\perp-\boldsymbol{x}'_\perp)\cdot\boldsymbol{p}_\perp}\,. \tag{7.24}$$

The sum over n is given by relation (7.19), leading to

$$
\begin{aligned}
& -\mathrm{i}\,\frac{E}{\pi}\,\mathrm{e}^{-\mathrm{i}\,\boldsymbol{x}_\parallel\cdot\boldsymbol{E}\cdot\boldsymbol{x}'_\parallel-\frac{E}{2}\|\boldsymbol{x}_\parallel-\boldsymbol{x}'_\parallel\|^2_\vartheta}\int_0^\infty \mathrm{d}s\int\frac{\mathrm{d}^2\boldsymbol{p}_\perp}{(2\pi)^2}\sum_{m=0}^\infty L_m\left(E\|\boldsymbol{x}_\parallel-\boldsymbol{x}'_\parallel\|^2_\vartheta\right)\\
\times\quad & \mathrm{e}^{-\mathrm{i}\,s\mu^2}\,\mathrm{e}^{-s4E(m+\frac{1}{2})}\,\mathrm{e}^{\,\mathrm{i}\,s\boldsymbol{p}_\perp^2-\mathrm{i}\,(\boldsymbol{x}_\perp-\boldsymbol{x}'_\perp)\cdot\boldsymbol{p}_\perp}\,,
\end{aligned}
\tag{7.25}
$$

and the resulting sum over m follows from equation (48.4.1) of [Han75]:

$$
\begin{aligned}
\mathrm{e}^{-y/2}\sum_{m=0}^\infty L_m(y)t^m &= \mathrm{e}^{-y/2}\frac{1}{1-t}\exp\left\{\frac{yt}{t-1}\right\}\\
&= \frac{1}{1-t}\exp\left\{\frac{y}{2}\frac{t^{1/2}+t^{-1/2}}{t^{1/2}-t^{-1/2}}\right\}\quad,\quad |t|<1\,,
\end{aligned}
\tag{7.26}
$$

which yields

$$
\begin{aligned}
& -\,\mathrm{i}\,\frac{E}{2\pi}\,\mathrm{e}^{-\mathrm{i}\,\boldsymbol{x}_\parallel\cdot\boldsymbol{E}\cdot\boldsymbol{x}'_\parallel}\int_0^\infty\mathrm{d}s\frac{1}{\sinh(2sE)}\\
\times\quad & \exp\left\{-\,\mathrm{i}\,s\mu^2-\frac{1}{2}E_\vartheta\|\boldsymbol{x}_\parallel+\boldsymbol{x}'_\parallel\|^2_\vartheta\coth(2sE)\right\}\int\frac{\mathrm{d}^2\boldsymbol{p}_\perp}{(2\pi)^2}\,\mathrm{e}^{\,\mathrm{i}\,s\boldsymbol{p}_\perp^2-\mathrm{i}\,(\boldsymbol{x}_\perp-\boldsymbol{x}'_\perp)\cdot\boldsymbol{p}_\perp}\,.
\end{aligned}
\tag{7.27}
$$

The integration over the momenta can be done using

$$
\int\mathrm{d}p\,\mathrm{e}^{\,\mathrm{i}\,sp^2-\mathrm{i}\,(x-y)p}=\sqrt{\frac{\mathrm{i}\,\pi}{s}}\,\mathrm{e}^{\,\mathrm{i}\,\frac{(x-y)^2}{4s}}
\tag{7.28}
$$

yielding

$$
\begin{aligned}
& \frac{E}{8\pi^2}\,\mathrm{e}^{-\mathrm{i}\,\boldsymbol{x}_\parallel\cdot\boldsymbol{E}\cdot\boldsymbol{x}'_\parallel}\int_0^\infty\frac{\mathrm{d}s}{s}\frac{1}{\sinh(s2E)}\\
\times\quad & \exp\left\{-\,\mathrm{i}\,s\mu^2-\frac{1}{2}E_\vartheta\|\boldsymbol{x}_\parallel-\boldsymbol{x}'_\parallel\|^2_\vartheta\coth(2sE)+\mathrm{i}\,\frac{(\boldsymbol{x}_\perp-\boldsymbol{x}'_\perp)^2}{4s}\right\}\,,
\end{aligned}
\tag{7.29}
$$

where the scalar products are understood to be Euclidean for the $\boldsymbol{x}_\perp$ components. Taking the limit $\epsilon\to 0$, thus $\vartheta\to\pi/2$, and substituting $E\to eE/2$ to conform to the conventions of [FGS91], this result is identical to equation (7.12) and proves the lemma. □

The eigenfunctions for the full regularized operator $\mathsf{D}_x^{(\epsilon)}$ factorize into components perpendicular to the electric field and the eigenfunctions of $(\mathsf{P}_\mu^2-\mu^2)_\epsilon$. Since the eigenvalues of the perpendicular momenta do not produce new singularities, we can neglect them in this calculation and also in the calculation leading to (7.12). Again they perfectly agree, extending this result to the two dimensional LSZ model at $\sigma=1$. We suspect that the ϑ-regularization leads to the causal propagators for $\sigma\neq 1$, too.

Note that the Schwinger parameter introduced in equation (7.23) only allows for the regularizations $\vartheta>0$ and $\mu^2-\mathrm{i}\,\epsilon$ because of the condition $\mathfrak{Im}(a)>0$, where the latter is usually associated to the Feynman boundary condition on the propagator. The other choices $\vartheta<0$ and $\mu^2+\mathrm{i}\,\epsilon$ can be applied using

$$
\frac{1}{a}=\mathrm{i}\int_{-\infty}^0\mathrm{d}s\;\mathrm{e}^{\,\mathrm{i}\,sa}\quad,\ \text{for}\ \mathfrak{Im}(a)<0\,.
\tag{7.30}
$$

The regularization $\mu^2+\mathrm{i}\,\epsilon$ is known as Dyson boundary condition, which leads to an anti-causal propagator, where anti-particles travel forward and particles backward in time. This suggests the conclusion that the regularization $\vartheta<0$ leads to the Dyson propagator.

The regularization of the mass $\mu^2\to\mathrm{e}^{-\mathrm{i}\,\epsilon}\mu^2$ is actually irrelevant for the analysis above. Its only function is to provide a continuous relation of the Minkowskian and the Euclidean wave operators with help of parameter ϑ alone, without the need to keep trace of additional minus signs in front of the mass term. This means that the interpretation in terms Feynman/Dyson propagator for the cases $\vartheta\to\vartheta\pm\pi/2$ still holds by regularizing just the operator P_μ^2.

The derivation of the propagator with help matrix basis may be compared to the calculation with other methods, such as Schwinger's derivation in his proper time formalism [Sch51], the "sum over solutions method" [FGS91] or the eigenvalue method using the continuous basis [Rit78]. Compared to the latter the matrix basis involves only polynomials and sums instead of the complicated integral expressions and thus brings along a strong simplification. As a further example how the matrix basis can be used serves the one-loop effective action of the same model as above. It has been calculated in appendix F. It is proposed that going beyond the constant field case might be possible using the ϑ-regularization and the matrix basis. This might help to probe QED in the non-perturbative regime (see e.g. [Rin01, HI09, Dun09, ILM10]). We conclude that the matrix basis may serve as a computational tool to simplify otherwise cumbersome calculations.

7.2 Continuous versus Matrix Basis

We now directly compare the continuous basis to the discrete matrix basis. The two-dimensional GW model in continuous basis with $\phi^{\star 3}$ interaction term at the self-dual point has been investigated in [Zah10]. We will give a short exposition of the aspects of this work with relevance for us, with its problems and possible solutions. The notation of [Zah10] compared to ours is such that $\theta = \lambda_{nc}^2$ and $E = 2/\lambda^2$. Its perturbation theory can be determined analogously to the matrix representation in the last section, with a different interaction vertex and propagator as demonstrated below. In 1+1 dimensions the continuous basis are the Wigner transformed tensor products

$$\chi_{st}^{k\ell}(\boldsymbol{x}) = \mathsf{W}\left[|\chi_s^k\rangle\langle\chi_t^\ell|\right](\boldsymbol{x}) \tag{7.31}$$

with $s, t = \pm$ and $k, \ell \in \mathbb{R}$. They can be represented in terms of confluent hypergeometric functions, but their exact form is irrelevant for the following. They satisfy

$$\begin{aligned} \mathsf{P}_\mu^2\, \chi_{st}^{k\ell}(\boldsymbol{x}) &= 4Ek\, \chi_{st}^{k\ell}(\boldsymbol{x}) \\ \tilde{\mathsf{P}}_\mu^2\, \chi_{st}^{k\ell}(\boldsymbol{x}) &= 4E\ell\, \chi_{st}^{k\ell}(\boldsymbol{x}) \end{aligned} \tag{7.32}$$

and obey the projector property

$$\chi_{st}^{k\ell} \star \chi_{s't'}^{k'\ell'} = \delta_{ts'}\delta(k'-\ell)\chi_{st'}^{k\ell'}\,. \tag{7.33}$$

The real fields expanded in terms of $\chi_{st}^{k\ell}$ read

$$\phi(\boldsymbol{x}) = \sum_{st}\int \mathrm{d}k\,\mathrm{d}\ell\, \chi_{st}^{k\ell}(\boldsymbol{x})\,\phi_{ts}^{\ell k} \tag{7.34}$$

with

$$\phi_{\ell k}^{ts} = \int \mathrm{d}^2\boldsymbol{x}\, \chi_{ts}^{\ell k}(\boldsymbol{x}) \star \phi(\boldsymbol{x})\,. \tag{7.35}$$

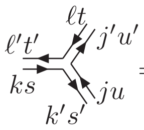
The GW wave operator takes the form

$$\left(\frac{1}{2}\mathsf{P}_\mu^2 + \frac{1}{2}\tilde{\mathsf{P}}_\mu^2 - \mu^2\right)_{st;t's'}^{k\ell;\ell's'} = (2E(k+\ell)-\mu^2)\delta_{ss'}\delta_{tt'}\delta(k-k')\delta(\ell-\ell')\,. \tag{7.36}$$

Due to zero eigenvalues this operator can not simply be inverted. In [Zah10] this problem is solved by adding the term $\mathrm{i}\,\epsilon\,\sigma_{st}(k,\ell)$ with some constant $\epsilon > 0$ and a sign function $\sigma_{st}(k,\ell)$. Depending on the explicit functional behavior of the sign function one gets different propagators. This function will be left undetermined for the time being such that the results may be compared with different propagators at the end. The double line notation is used with the propagator given by

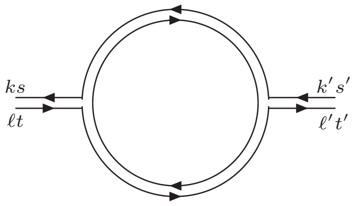
$$\begin{array}{c} \ell t \quad \ell' t' \\ \overleftarrow{\overrightarrow{\quad\quad}} \\ ks \quad k's' \end{array} = \frac{-1}{2E(k+\ell)-\mu^2+\mathrm{i}\,\epsilon\,\sigma_{st}(k,\ell)}\;\delta(k-k')\delta(\ell-\ell')\delta_{ss'}\delta_{tt'}$$

and the vertex given by



$$= \mathrm{i}\, g\, \delta_{ss'} \delta_{tt'} \delta_{uu'} \delta(k-k')\delta(\ell-\ell')\delta(j-j')\,,$$

with coupling constant g. It follows that the planar fish graph



is given by

$$\begin{aligned} & g^2 \delta(k-k')\delta(\ell-\ell')\delta_{ss'}\delta_{tt'} \\ & \times \sum_u \int \mathrm{d}j\,\mathrm{d}j' \frac{1}{4E(k+j)-\mu^2+\mathrm{i}\,\epsilon\,\sigma_{su}(k,j)} \frac{1}{4E(j'+\ell)-\mu^2+\mathrm{i}\,\epsilon\,\sigma_{tu}(j',\ell)} [\delta(j-j')]^2\,. \end{aligned} \tag{7.37}$$

This expression is divergent due to the squared δ-function coming from the undetermined loop integration. It is no UV divergence in the usual sense, as it occurs before performing loop integrals, and shows up in every $\phi^{\star n}$ theory with $n \geq 3$ for graphs with an unbroken internal line. A possible cure for this divergence is a box regularization. Instead of using the $\mathrm{i}\,\epsilon$ regularization one puts the system into a box with finite volume and imposes periodic boundary conditions. Instead of a continuous spectrum we get a discrete one leading to Kronecker δ-functions and sums instead of Dirac δ-functions and integrals. Obviously this procedure renders this diagram finite. However, the box regularization is an IR cutoff, which is likely to destroy the LS-covariance at quantum level unless one imposes in addition a suitable UV-cutoff. In contrast, the regularized matrix approach has the same effect on the vertex functions as the box regularization, but at the same time keeps the model LS-covariant, as will be demonstrated in the next section.

7.3 LS-Duality at Quantum Level

The ϑ-regularization allows us to regularize the LS-covariant theories such that the LS-duality is preserved at quantum level. This is done in the same spirit as in section 4.5 with the ϑ-regularization being a new ingredient. In the following this will be demonstrated for the two-dimensional GW model. The general LSZ case is exactly the same.

An important question is, how the ϑ-regularization affects the behavior under LS-duality. The regularized propagator with $\vartheta = \pi/2 - \epsilon > 0$ reads

$$\begin{aligned} \Delta^{(\epsilon)}(\boldsymbol{x},\boldsymbol{x}') &= \langle \boldsymbol{x}| \left(\frac{1}{2}\mathsf{P}_\mu^2 + \frac{1}{2}\tilde{\mathsf{P}}_\mu^2 - \mu^2\right)^{-1}_\epsilon |\boldsymbol{x}'\rangle \\ &= \sum_{m,n} \frac{f_{mn}^{(E_\vartheta)}(\boldsymbol{x}) f_{nm}^{(E_\vartheta)}(\boldsymbol{x}')}{2\,\mathrm{i}\,E\,(m+n+1) - \mathrm{e}^{-\mathrm{i}\,\epsilon}\mu^2}\,. \end{aligned} \tag{7.38}$$

In appendix H we show that the Fourier transformation of matrix functions is given by

$$\mathcal{F}[f_{mn}^{(E_\vartheta)}](\boldsymbol{k}) = f_{nm}^{(1/E_\vartheta)}(\boldsymbol{k}) = \frac{(-\mathrm{i}\,)^{m-n}}{E} f_{mn}^{(E_\vartheta)}(\tilde{\boldsymbol{k}}) \tag{7.39}$$

with $\tilde{\boldsymbol{k}} = \boldsymbol{E}^{-1} \cdot \boldsymbol{k} = -E^{-1}(k^1, k^0)$.[1] Since

$$\mathcal{F}[(\mathsf{P}^2(\vartheta) + \tilde{\mathsf{P}}^2(\vartheta)) f_{mn}^{(E_\vartheta)}](\boldsymbol{k}) = 4E_\vartheta\,(m+n+1)\,\mathcal{F}[f_{mn}^{(E_\vartheta)}](\boldsymbol{k})\,, \tag{7.40}$$

we find that Fourier transformation relates the propagator in position space to the momentum space propagator even in the regularized case:

$$\hat{\Delta}^{(\epsilon)}(\boldsymbol{k}, \boldsymbol{k}') = \frac{1}{E^2} \Delta^{(\epsilon)}(\tilde{\boldsymbol{k}}; \tilde{\boldsymbol{k}}')\,. \tag{7.41}$$

Analogously to the Euclidean case the UV/IR-regularization now amounts to cutting off the sums at some finite N by modifying the regularized position space propagator as

$$\Delta_\Lambda^{(\epsilon)}(\boldsymbol{x}, \boldsymbol{x}') = \langle \boldsymbol{x}| \left(\frac{1}{2}\mathsf{P}_\mu^2 + \frac{1}{2}\tilde{\mathsf{P}}_\mu^2 - \mu^2\right)_\epsilon^{-1} L\left(\Lambda^{-2}|\mathsf{P}^2(\vartheta) + \tilde{\mathsf{P}}^2(\vartheta)|\right) |\boldsymbol{x}'\rangle\,, \tag{7.42}$$

where $\Lambda \in \mathbb{R}_+$ is a cut-off parameter and L a smooth cut-off function which is monotonically decreasing, with $L(y) = 1$ for $y < 1$ and $L(y) = 0$ for $y > 2$. We adjust the matrix functions as to diagonalize the LSZ propagator

$$\begin{aligned} \Delta_{\Lambda, mn;k\ell}^{(\epsilon)} &= \int_x f_{mn}^\epsilon(\boldsymbol{x}) \left(\frac{1}{2}\mathsf{P}_\mu^2 + \frac{1}{2}\tilde{\mathsf{P}}_\mu^2 - \mu^2\right)_\epsilon^{-1} L\left(\Lambda^{-2}|\mathsf{P}^2(\vartheta) + \tilde{\mathsf{P}}^2(\vartheta)|\right) f_{k\ell}^\epsilon(\boldsymbol{x}) \\ &= \frac{\delta_{m\ell}\,\delta_{nk}}{2\,\mathrm{i}\,E(m+n+1) - \mathrm{e}^{-\mathrm{i}\,\epsilon}\mu^2} L\left(\Lambda^{-2} 4E(m+n+1)\right)\,. \end{aligned} \tag{7.43}$$

The interaction vertices in matrix representation are now quite complicated, being proportional to

$$\int_x \left(f_{m_1n_1}^\epsilon \star_\theta f_{m_2n_2}^\epsilon \star_\theta f_{m_3n_3}^\epsilon \star_\theta f_{m_4n_4}^\epsilon\right)(\boldsymbol{x}) \tag{7.44}$$

with $\theta \neq 2/E$ in general. Since for $\epsilon > 0$ the f_{mn}^ϵ are in $\mathcal{S}_\alpha^\alpha(\mathbb{R}^2)$ with $\alpha \geq 1/2$, which is closed with respect to the star-product, the interaction vertex (7.44) is well-defined. Feynman diagrams can now be produced by suitable derivatives with respect to the external sources involving the regularized propagator. Denoting

$$\Delta_{\Lambda, mn;k\ell}^{(\epsilon)} = \delta_{mk}\,\delta_{n\ell}\,C_\Lambda^{(\epsilon)}(m, n)\,, \tag{7.45}$$

they have the schematical form

$$\sum_{n_1, m_1, \ldots, n_K, m_K = 0} \prod_{k=1}^K C_\Lambda^{(\epsilon)}(m_k, n_k)(\cdots)\,, \tag{7.46}$$

where $(\cdots)$ denotes the contributions from the noncommutative interaction vertices and combinatorial factors. Since the propagator is nonzero only if $4E(m_k + n_k + 1) < 2\Lambda$, which at finite Λ is true solely for a finite number of distinct values of $(m, n) \in \mathbb{N}_0^2$, every Feynman amplitude is represented by a finite sum and thus constitutes well-defined Green functions in the matrix basis circumventing the problem of the right test function space for the time being. By multiplying these expression with $f_{m_in_i}^\epsilon(\boldsymbol{x}_i)$ for $i = 1, \ldots, M$ and M the number of external vertices, we get back the position space Green functions by summing over all m_i, n_i. They are also well-defined, since they are build by finite sums of well-defined objects. This establishes the quantum duality in Minkowski spacetime for the case $\epsilon > 0$.

To prove the duality at $\epsilon = 0$ in the same manner as above, one has to ensure that the interaction vertex away from the dual point is well-defined, which is not obviously true. We conclude that, to be on the safe side, the ϑ-regularization should be kept unless the matrix cutoff has been removed and all summations and integrations have been performed.

[1] Note that there is a subtle difference between the Euclidean and Minkowskian case. Contrary to the ordinary Landau case in Euclidean space, the (not yet rescaled) Fourier transformed generalized Landau functions have swapped indices and an inverted regularization parameter $\vartheta \to -\vartheta$. The former is equivalent to an inversion of time (or space) while the latter corresponds to an interchange of particles and anti-particles. This follows from the results of section 7.1.1, where the regularization $\vartheta > 0$ has been identified with the Feynman boundary condition and $\vartheta < 0$ with the Dyson boundary condition. The specific rescaling in both cases, which are formally identical but differ by the metric which is used, compensates for this difference.

As is usually the case, the limit $\Lambda \to \infty$ may still be ill-defined and may require a renormalization. In addition, the results from section 6.2 are not able to exclude that even at finite $\epsilon > 0$ there might be extra divergences at $\Lambda \to \infty$ if we work in the matrix basis, stemming from the generalized matrix basis itself. This, however, does not affect the LS-covariance of the theory, which has been achieved for the Green's functions in position space through a regularization of the propagators in equation (7.42). This result is independent of the matrix basis.

7.4 Unitarity

The unitarity of the scattering matrix is one of the main pillars of commutative quantum field theory, in which statements as analyticity, microscopic causality and unitarity are roughly interchangeable. These concepts are expected to be disentangled in NCQFT due to the lack of locality and Lorentz invariance. In [GM00] new singularities in the correlation functions of the usual $\phi^{\star 3}$ and $\phi^{\star 4}$-theories in the standard perturbative setup have been observed, which imply a violation of the cutting rules and thus the breakdown of unitarity. The question arises what happens to the analytical structure, if the NCQFT is put into a background electromagnetic field, making the theory LS-duality covariant? The interesting new features of the LS-covariant models in Minkowski spacetime are the duality between Θ and E and the vacuum instability. Thus, there are new singularities due to pair creation even to zeroth order in the coupling g.

We shortly review the main aspects of unitarity starting from the Hamiltonian formalism in commutative quantum field theory with stable vacuum. For simplicity we consider a theory with one species of real bosons without external field. In a scattering experiment, an initial state $|i, in\rangle$ is assumed to consist of free particles in the remote past. The state evolves in time in the presence of some interaction, while in the far future the detectors are set up to detect a state $|f, out\rangle$, consisting of free particles of definite momenta and maybe other quantum numbers. It will thereby assumed that the asymptotic initial and final Hilbert spaces may be constructed as free particle Fock spaces, with ladder operators $a_{\boldsymbol{p}}(in), a^\dagger_{\boldsymbol{p}}(in)$ and $a_{\boldsymbol{p}}(out), a^\dagger_{\boldsymbol{p}}(out)$ corresponding to particles with definite momenta acting on unique vacua $|0, in\rangle$ and $|0, out\rangle$, respectively. Since the theory has a stable vacuum, these two spaces are equivalent with $|0, in\rangle = |0, out\rangle$ up to a phase.

In the following we will mainly work in the interaction picture. For those few times we need to switch to the the Heisenberg picture we will designate the states with a subscript H. In the interaction picture the two Hilbert spaces are related by a unitary operator, the S-matrix operator $\hat{S}$ with

$$\hat{S}^\dagger a_{\boldsymbol{p}}(in)\hat{S} = a_{\boldsymbol{p}}(out) \quad , \quad \hat{S}^\dagger a^\dagger_{\boldsymbol{p}}(in)\hat{S} = a^\dagger_{\boldsymbol{p}}(out)$$

and

$$\begin{aligned} \hat{S}|\boldsymbol{p}_1, \dots, \boldsymbol{p}_n; in\rangle &= |\boldsymbol{p}_1, \dots, \boldsymbol{p}_n; out\rangle \\ \langle \boldsymbol{p}_1, \dots, \boldsymbol{p}_n; out|\hat{S} &= \langle \boldsymbol{p}_1, \dots, \boldsymbol{p}_n; in| \end{aligned} \tag{7.47}$$

up to an irrelevant phase. The probability of the process to take place is given by the S-matrix element

$$S_{fi} = {}_H\langle f, out|i, in\rangle_H = \langle f, in|\hat{S}|i, in\rangle \,. \tag{7.48}$$

S_{fi} is related to the n-point functions in a specific way, prescribed by the LSZ reduction formula. As an example we consider the scattering process $\boldsymbol{k}_1, \boldsymbol{k}_2 \to \boldsymbol{p}_1, \boldsymbol{p}_2$, where the initial and final states have definite momenta $\boldsymbol{p}_i$ and $\boldsymbol{k}_i$. The corresponding S-matrix element reads

$$\begin{aligned} {}_H\langle \boldsymbol{p}_1, \boldsymbol{p}_2|\boldsymbol{k}_1, \boldsymbol{k}_2\rangle_H \;=\;& \text{disconnected terms} \\ &+ \left(\frac{\mathrm{i}}{\sqrt{Z}}\right)^4 \int_{y_1}\int_{y_2}\int_{x_1}\int_{x_2} u^*_{\boldsymbol{p}_1}(\boldsymbol{y}_1)u^*_{\boldsymbol{p}_2}(\boldsymbol{y}_2)\overrightarrow{(\partial^2_{y_1} + m^2)}\overrightarrow{(\partial^2_{y_2} + m^2)} \\ &\times\; G^{(4)}(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{y}_1, \boldsymbol{y}_2)\overleftarrow{(\partial^2_{x_1} + m^2)}\overleftarrow{(\partial^2_{x_2} + m^2)}u_{\boldsymbol{k}_1}(\boldsymbol{x}_1)u_{\boldsymbol{k}_2}(\boldsymbol{x}_2) \end{aligned} \tag{7.49}$$

where m is the *physical mass* and Z the field strength renormalization. The eigenstates $u_{\boldsymbol{p}}(x)$ and $u^*_{\boldsymbol{p}}(x)$ are Klein-Gordon *in*-states and *out*-states, respectively, with definite momentum $\boldsymbol{p}$, energy $\omega_{\boldsymbol{p}}$ and

$$u_{\boldsymbol{p}}(x) = \frac{1}{\sqrt{(2\pi)^D 2\omega_{\boldsymbol{p}}}}\,\mathrm{e}^{-\mathrm{i}\,\boldsymbol{p}\cdot\boldsymbol{x}} \quad , \quad u^*_{\boldsymbol{p}}(x) = \frac{1}{\sqrt{(2\pi)^D 2\omega_{\boldsymbol{p}}}}\,\mathrm{e}^{+\mathrm{i}\,\boldsymbol{p}\cdot\boldsymbol{x}} \,. \tag{7.50}$$

The four-point function $G^{(4)}$ can be expressed in the interaction picture as

$$G^{(4)}(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{y}_1, \boldsymbol{y}_2) = \frac{\langle 0|T\,(\hat{S}\,\hat{\phi}(\boldsymbol{x}_1)\hat{\phi}(\boldsymbol{x}_2)\hat{\phi}(\boldsymbol{y}_1)\hat{\phi}(\boldsymbol{y}_2))|0\rangle}{\langle 0|S|0\rangle} \tag{7.51}$$

with $\hat{\phi}$ the field operators in the interaction picture and T the time ordering operator. The S-matrix operator can be written as

$$\hat{S} = T \exp\left(-\,\mathrm{i}\int \mathrm{d}t\, \hat{H}_I(t)\right) \tag{7.52}$$

with $\hat{H}_I(t)$ being the interaction Hamiltonian in the interaction picture. The perturbative expansion of the S-matrix operator (7.52) plus Wick's contraction theorem leads to a perturbative evaluation of this expression in terms of Feynman diagrams.

Note that the n-point function does not know about which particle is incoming and which is outgoing. This designation is imposed by projecting onto the respective eigenfunctions, which in the Klein-Gordon case amounts to fixing the signs of the external momenta. At this step pair creating processes are excluded through δ-functions caused by translation invariance. Disconnected vacuum graphs factorize from all graphs into a phase factor which is identical to $\langle 0|S|0\rangle$ and thus get canceled by the normalization factor of the n-point function. Self-energy subgraphs connected to the external propagators, like the tadpole or the fish-graph, simply turns the free external propagators into the full "interaction propagators" with shifted mass pole and alternated residue. This corresponds to a mass and field strength renormalization. This has been taken into account already in (7.49) by usage the physical mass m and by the insertion a factor $Z^{-1/2}$ for each external propagator. In Fourier space the probability amplitude is proportional to

$$(\boldsymbol{p}_1^2 - m^2)(\boldsymbol{p}_2^2 - m^2)(\boldsymbol{k}_1^2 - m^2)(\boldsymbol{k}_2^2 - m^2)\hat{G}^{(4)}(\boldsymbol{p}_1, \boldsymbol{p}_2, -\boldsymbol{k}_1, -\boldsymbol{k}_2)\,, \tag{7.53}$$

with $\hat{G}^{(4)}$ being the Fourier transformed four-point function. The momenta $\boldsymbol{k}_i$ and $\boldsymbol{p}_i$ are "on mass-shell" and would force the whole expression to vanish, if the n-point function had no poles at $\boldsymbol{p}_1^2 = \boldsymbol{p}_2^2 = \boldsymbol{k}_1^2 = \boldsymbol{k}_2^2 = m^2$. One can show that this is not the case and the factors cancel exactly the full interacting external propagator of each diagram, which are now called *amputated* diagrams. One disregards the case of no actual scattering by splitting up the T-matrix

$$\langle f|\hat{S}|i\rangle = \langle f|i\rangle - \mathrm{i}\,\langle f|\hat{T}|i\rangle\,, \tag{7.54}$$

where for translation invariant theories $\langle f|\hat{T}|i\rangle$ is proportional to an overall δ-function imposing energy-momentum conservation. In summary, the S-matrix elements are given by all connected and amputated Feynman diagrams, which in turn may be evaluated in the usual way using Feynman rules.

Unitarity of the S-matrix now formally reads

$$\hat{S}\hat{S}^\dagger = \hat{S}^\dagger\hat{S} = \mathbb{1} \tag{7.55}$$

and implies relations between different transition probabilities. Sandwiching this relation between in- and out-states with $i = f$ and inserting a complete set of asymptotic states $|n\rangle$, we find

$$2\mathfrak{Im}\langle i|\hat{T}|i\rangle = -\sum_n |\langle n|\hat{T}|i\rangle|^2\,, \tag{7.56}$$

which is known as *optical theorem*. Verification of relation (7.56) for single processes $|i\rangle$ is thus a test for the unitarity of the theory. The usual approach to verify (7.56) is to use the *cutting rules* to the corresponding Feynman diagrams, which is as follows. A given graph consists of combinations of Feynman propagators $(p^2 - m^2 + \mathrm{i}\,\epsilon)^{-1}$ and constant vertices. It possess an imaginary part because of

$$\lim_{\epsilon\to\infty} \frac{1}{x + \mathrm{i}\,\epsilon} = P\frac{1}{x} - \mathrm{i}\,\pi\delta(x)\,. \tag{7.57}$$

The exact value of the imaginary part of a given diagram may be obtained as follows: drawing lines through internal propagators such that the Feynman diagram splits up into two pieces, followed by a replacement of each cut propagator $(p^2 - m^2 + \mathrm{i}\,\epsilon)^{-1}$ through $-2\pi\,\mathrm{i}\,\delta(p^2 - m^2)$. This should be done in all possible ways, where the imaginary part of the original graph is the sum of all contributions coming from the cut diagrams.

In [GM00], Gomis and Mehen checked the cutting rules for the two point function in $\phi^{\star 3}$-theory and four-point function in $\phi^{\star 4}$-theories to second order in perturbation theory. Since the theories are still translation invariant and the propagators are identical to the commutative case, the above procedure to find the imaginary part of a given Feynman diagram also applies in noncommutative QFT. However, it was found that the diagrams have additional branch cuts along

$$p \circ p = -p_\mu \Theta^{\mu\nu} \Theta_{\nu\sigma} p^\sigma \leq 0\,, \tag{7.58}$$

which are accessible for time/space noncommutative theories, and in these cases cause a violation of unitarity. They resemble particle-production cuts. As has already been pointed out at the end of section 2.2.3, the reason for this curiosity is the wrong application of Wick's theorem in order to reduce the determination of n-point functions to the evaluation of Feynman diagrams, which is not allowed since time derivatives and time ordering do not commute [BDFP02, Bah04]. For the two-point function at second order of the perturbative expansion in a $\phi^{\star 3}$-theory the non-unitarity has been spotted to the nonvanishing of the terms

$$\Delta_{ret} \star \Delta_{av} + \Delta_{av} \star \Delta_{ret} \neq 0 \tag{7.59}$$

for time/space-noncommutativity, which in turn is due to $\theta \star \theta \neq \theta$. Here Δ_{av} and Δ_{ret} are the advanced and retarded propagators, respectively. The question arises, if it is possible to retain unitarity in some way, but at the same time keep the Feynman diagrams as the main building blocks of the perturbative expansion. In [AGBZ01] the singularities have been further investigated and assigned to the production of tachyonic states. By adding new states to the Hilbert space the cutting rules are formally fulfilled, however, unitarity is still absent due to the presence of tachyonic states in the asymptotic Hilbert space. What happens in the LS-covariant case?

To give an answer, we first describe the situation for *commutative* theories where pair creation is allowed. A typical example is the usual QED in a vacuum stability violating external field. The following exposition is quite general and applies to complex scalars and spinors. We will leave aside the technical subtleties and give only a sketchy overview of the general proceeding in these cases. For an extensive overview see [FGS91]. We start with an heuristic argument, assuming the asymptotic Hilbert spaces may be constructed as before as Fock spaces. Due to the pairs which are created from the vacuum in the course of time, the probability for an initial vacuum to stay the vacuum is not equal to one:

$$|{}_H\langle 0, out | 0, in \rangle_H| < 1\,. \tag{7.60}$$

Going to the interaction picture, where the field operators now fulfill the equation of motion of the particles moving in the *external* field, the vacuum-to-vacuum probability is given by

$$|{}_H\langle 0, out | 0, in \rangle_H| = |\langle 0, out | \hat{S} | 0, in \rangle| < 1 \tag{7.61}$$

with $\hat{S}$ the S-matrix operator (7.52). We follow that in contrast to the relations (7.47) of the ordinary case, $\hat{S}|0, in\rangle \neq |0, out\rangle$ and $\langle 0, out|\hat{S} \neq \langle 0, in|$. The S-matrix element for an arbitrary process $|i, in\rangle \to |f, out\rangle$ is defined similarly by

$$S_{fi} = {}_H\langle f, out | i, in \rangle_H = \langle f, out | \hat{S} | i, in \rangle \tag{7.62}$$

where we find contrary to (7.48) again an out-state to the left of the S-matrix operator. Thus the correlation function can not be obtained by a reduction to normal form relative to one vacuum, but demands a reduction to a generalized normal form relative to the two vacua $\langle 0, out|$ and $|0, in\rangle$. How to do this will be sketched now.

On the level of solutions of the equation of motion, pair production manifests itself in an inevitable mixing of positive and negative frequencies. This means that solutions which have definite positive or negative frequency throughout all times are no longer available. Since those are necessary for the ordinary canonical quantization scheme to apply one proceeds as follows [Git77]. One constructs two complete and orthogonal sets of solutions of the energy eigenvalue equation, one at each of the two finite time instances t_{in} and t_{out}. At these time instances, these sets can be split into positive and negative frequency solutions and canonical quantization of the fields applies as usual by quantizing the positive/negative energy solutions in terms of ladder operators, which act on the respective Fock vacua. The limits $t_{in} \to -\infty$ and $t_{out} \to \infty$ are taken afterwards, such that the solutions remain their character as positive/negative energy solutions and the relations which characterize pair production processes (equations (7.64)-(7.71) below) remain well-defined.[2] The vacua $|0, in\rangle$ and $|0, out\rangle$ now differ from each other, as well as the ladder operators $a_n(in), a_n^\dagger(in), b_n(in), b_n^\dagger(in)$ representing particles/antiparticles of definite momenta at t_{in} which are different to $a_n(out), a_n^\dagger(out), b_n(out), b_n^\dagger(out)$ representing particles/antiparticles of definite momenta at t_{out}. The index n thereby compactly designates all the quantum numbers as momentum and spin. The usual commutation relations hold among the in-operators and among the out-operators, as well as

$$\begin{aligned} a_n(in)|0, in\rangle &= b_n(in)|0, in\rangle = 0 \\ a_n(out)|0, out\rangle &= b_n(out)|0, out\rangle = 0\,. \end{aligned} \tag{7.63}$$

for all n. The mixing of frequencies imply relations among the ladder operators at different times

$$a_m(in) = \sum_n G\,({}_+|{}^+)_{mn}\, a_n(out) + \sum_n G\,({}_+|{}^-)_{mn}\, b_n^\dagger(out)\,, \tag{7.64}$$

$$b_m(in) = \sum_n G\,({}^+|{}_-)_{mn}\, a_n^\dagger(out) + \sum_n G\,({}^-|{}_-)_{mn}\, b_n(out)\,, \tag{7.65}$$

$$a_m(out) = \sum_n G\,({}^+|{}_+)_{mn}\, a_n(in) + \sum_n G\,({}^+|{}_-)_{mn}\, b_n^\dagger(in)\,, \tag{7.66}$$

$$b_m(out) = \sum G\,({}_+|{}^-)_{mn}\, a_n^\dagger(in) + \sum_n G\,({}_-|{}^-)_{mn}\, b_n(in)\,, \tag{7.67}$$

where the *Bogoliubov-coefficients* $G(\cdot)$ are a measure for particle production. A stable vacuum thus implies $G\,({}^\pm|{}_\mp) = G\,({}_\pm|{}^\mp) = 0$ with all others being equal to unity. A generalized Wick theorem with respect to $\langle 0, out|$ and $|0, in\rangle$ can be realized by expressing all operators in terms of $a_n(in), b_n(in), a_n^\dagger(out), b_n^\dagger(out)$ alone. The procedure is then to pull all creation operators to the left of the annihilation operators such that the relations (7.63) apply as in the ordinary case. The occurring generalized contractions may be obtained by exploiting the usual commutation relations and equations (7.64)-(7.67):

$$\begin{aligned} \underbrace{a_m(out)a_n^\dagger}(in) &= G^{-1}\,({}_+|{}^+)_{mn} &&:= \omega(m^+|n^+) && (7.68)\\ \underbrace{b_m(out)b_n^\dagger}(in) &= G^{-1}\,({}^-|{}_-)_{mn} &&:= \omega(m^-|n^-) && (7.69)\\ \underbrace{a_m(out)b_n}(out) &= \left[\textstyle\sum_k G^{-1}\,({}_+|{}^+)_{mk}\, G\,({}_+|{}^-)_{kn}\right]_{mn} &&:= \omega(m^+n^-|0) && (7.70)\\ \underbrace{b_n^\dagger(in)a_n^\dagger}(in) &= \left[\textstyle\sum_k G\,({}_-|{}^+)_{mk}\, G^{-1}\,({}_+|{}^+)_{kn}\right]_{mn} &&:= \omega(0|m^-n^+) && (7.71) \end{aligned}$$

While $\omega(m^+|n^+)$ and $\omega(m^-|n^-)$ are the relative probabilities of particles and anti-particles to be scattered by the external field (compare equation (7.4) for the causal propagator), the quantities $\omega(m^+n^-|0)$ and $\omega(0|m^-n^+)$ measure the relative probabilities for pair creation and pair annihilation in the vacuum.

By expressing the field operators $\hat{\phi}$ in terms of the ladder operators, the S-matrix element

$$S_{fi} = \langle f, out|\hat{S}|i, in\rangle \tag{7.72}$$

can be now be calculated using the generalized Wick contractions. The matrix element may be obtained as in equation (7.49), by substituting the Klein-Gordon operators by the wave operators of the model and $u_{\boldsymbol{p}}(\boldsymbol{x})$ and $u^*_{\boldsymbol{p}}(\boldsymbol{x})$ by the new *in*- and *out*-states of definite momenta. The S-matrix element is then the amputated correlation function projected on these initial and final momentum states. Rearranging all creation operators to the left of the annihilation operators, the correlation function can be expressed in terms of the usual Feynman diagrams with the causal propagator given by

$$\mathrm{i}\,\Delta_c(\boldsymbol{x}, \boldsymbol{y}) = \frac{\langle 0, out|T\hat{\phi}(\boldsymbol{x})\hat{\phi}^\dagger(\boldsymbol{y})|0, in\rangle}{\langle 0, out|0, in\rangle}\,, \tag{7.73}$$

which may be calculated by one of the methods outlined in section 7.1. S_{fi} is still a sum of a unit matrix and a T-matrix, where the overall δ-function in front of the scattering part of (7.72) is absent due to the lack of energy-momentum conservation. New transition channels must be taken into account, corresponding to pairs created from the vacuum.

[2] Conditions on the solutions such that the Fock spaces exist may be found in [Git77].

Apart from this, the usual cutting rules no longer hold. Demanding unitarity of the S-matrix as in equation (7.54), we find the generalization of the optical theorem

$$2\mathfrak{Im}\langle i, in|\hat{T}|i, in\rangle = -\sum_n |\langle n, out|\hat{T}|i, in\rangle|^2 \,. \tag{7.74}$$

The contraction scheme discussed above leads to the usual Feynman diagrams on the rhs with additional pair production diagrams. However, on the lhs one has diagrams involving the propagator

$$\mathrm{i}\,\Delta_c^{in}(\boldsymbol{x},\boldsymbol{y}) = \langle 0, in|T\hat{\phi}(\boldsymbol{x})\hat{\phi}^\dagger(\boldsymbol{y})|0, in\rangle \tag{7.75}$$

instead of the Feynman propagator. These two propagators do not coincide in presence of an instable vacuum. In order to check unitarity of the LS-covariant theories, one has to find a relation between Δ_c^{in} and Δ_c. Quite generally, by decomposing the state $\langle 0, in|$ in a complete set of out-states one finds

$$\Delta_c^{in}(\boldsymbol{x},\boldsymbol{y}) = \Delta_c(\boldsymbol{x},\boldsymbol{y}) + \Delta^a(\boldsymbol{x},\boldsymbol{y})\,, \tag{7.76}$$

$$\Delta^a(\boldsymbol{x},\boldsymbol{y}) = -\mathrm{i}\sum_{mn}\omega(m^-n^+|0)^*\langle 0, out|a_m(out)b_n(out)T\hat{\phi}(\boldsymbol{x})\hat{\phi}^\dagger(\boldsymbol{x})|0, in\rangle\,, \tag{7.77}$$

where Δ^a is a solution of the equation of motion. It should be noted that the usage of the in-propagator can not be circumvented by considering $\langle i, out|$ instead of $\langle i, in|$ in equation (7.74). This would lead to the relation

$$\mathrm{i}\,\langle i, out|(T^\dagger - T)|i, in\rangle = -\sum_n\langle 0, out|T|n, in\rangle\langle n, in|T^\dagger|0, in\rangle \tag{7.78}$$

$$= -\sum_n\langle 0, out|T|n, out\rangle\langle n, out|T^\dagger|0, in\rangle\,. \tag{7.79}$$

Since

$$\langle i, out|T|i, in\rangle^* = \langle i, in|T^\dagger|i, out\rangle\,, \tag{7.80}$$

the lhs of equation (7.78) is not the imaginary part of $\langle i, out|T|i, in\rangle$. In addition, on the rhs we find Feynman diagrams involving the propagator Δ_c^{in} or

$$\mathrm{i}\,\Delta_c^{out}(x,y) = \langle 0, out|T\hat{\phi}(x)\hat{\phi}^\dagger(y)|0, out\rangle \tag{7.81}$$

depending on whether we insert a complete set of in-states (7.78) or out-states (7.79).

Now we come to the LS-covariant theories, where in addition to the instable vacuum we have a noncommutative interaction term. The interaction Hamiltonian is symmetric such that formally the S-matrix is unitary in the Hamiltonian formalism [Bah04]. As pointed out above, the proof for Wick's theorem does not apply anymore if the interaction is nonlocal in time, which is also true for the generalized contraction theorem. Thus unless there are some "magic cancellations" the perturbative quantum theory based on modified Feynman rules will lead to a non-unitary S-matrix. However, this may happen and has to be checked. But even in the case of unitarity violation it is interesting to see how the non-unitarity violating terms look like, and whether there is a possibility to retain unitarity by modifying the theory.

A first attempt towards an answer to the unitarity issue for LS-covariant theories in the standard perturbation setup was made in [Zah10] for the self-dual GW model with $\phi^{\star 3}$ interaction. The imaginary parts of the contributions to the two-point function at second order in perturbation theory have been calculated and compared to the expressions obtained from the cutting rules. The propagators which are used are determined via $\mathrm{i}\,\epsilon$ prescription:

$$\Delta_{ks,k's'}^{\ell t,\ell' t'} = \frac{-1}{2E(k+\ell) - \mu^2 \pm \mathrm{i}\,\epsilon}\delta(k-k')\delta(\ell-\ell')\delta_{ss'}\delta_{tt'}\,. \tag{7.82}$$

Thus the quantity which has been calculated is not the imaginary part of the Feynman diagrams corresponding to $\langle i, in|T^{(2)}|i, in\rangle$ but corresponding to $\langle i, out|T^{(2)}|i, in\rangle$. These results have, however, then been compared to diagrams coming from the rhs of (7.74), where the mismatch has been interpreted as a lack of unitarity. From the discussion above we find that for a correct investigation we need the probability for pair production $\omega(\cdot|0)$, in order to relate the propagators Δ_c and Δ_c^{in}. Since for the models under consideration, the free LSZ and GW model, these are unknown to the author, we cannot give a satisfactory answer at this point and leave this issue for a future investigation.

8 Renormalization of the LS-Covariant Models

One of the most intriguing features of Euclidean LS-covariant models is their renormalizability. We will not prove here the renormalizability of their Minkowskian counterparts, but start this program by deriving their propagators in position and matrix representation. First, we give a brief account of the methods which were successfully used in Euclidean space. After determining the propagators we will shortly discuss their asymptotics.

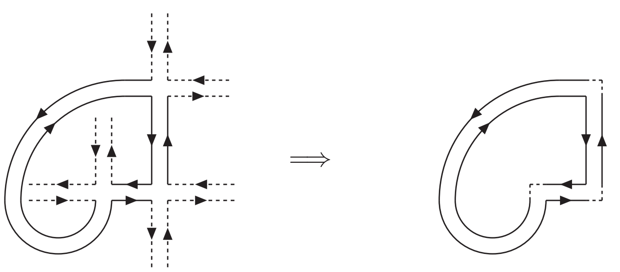
8.1 Multiscale Analysis

Multiscale analysis has been used to prove renormalizability of the LSZ, GW, vGN model and the translation invariant model. Though in their original proof Grosse and Wulkenhaar used Polchinski's RG equation, we will introduce the multiscale analysis in order to explain the relevant steps towards the renormalization of LS-covariant models in Minkowski spacetime. Multiscale analysis is independent of the precise representation of the model and has been successfully applied to both position- and matrix space. Multiscale analysis replaces the sharp cutoffs in matrix space by smoother ones directly in the Schwinger parameter representation of the propagator. For a general account of this method see [Riv91, Riv07b].

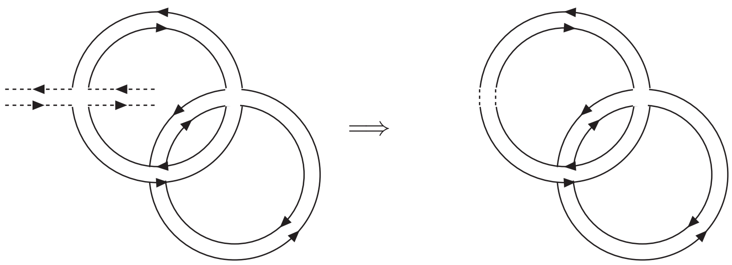
We will now give a sketchy illustration of how the asymptotic behavior of the propagator are used to prove the renormalizability of the GW model in Euclidean space following [RVTW06, Riv07b]. Feynman graphs for matrix models are written using the double line formalism. These graphs can not be drawn on a plane, but on two-dimensional Riemann surfaces with non-trivial topological structure. The power counting of a matrix model depends essentially on this topological data. Let G be a graph with V vertices, I internal (double) lines and F faces. To get F one has to amputate the external legs. Then F is the number of closed single lines and B the number of those closed lines which carry external legs. The Euler characteristic of the Riemann surface defined by these graphs is given by

$$\chi = 2 - 2g = V - I + F \tag{8.1}$$

which defines the genus g of the manifold. The genus g and the number B are a measure for non-planarity. As an illustration how the topological data of a ribbon graph can be determined serve the following examples:



$$\left.\begin{array}{l} V=3 \\ I=3 \\ F=2 \\ B=2 \end{array}\right\} \Longrightarrow g=0$$



In the Grosse-Wulkenhaar model the N-leg ribbon graph in four dimensions has the power counting degree

$$\omega(G) = (4 - N) - 4(2g + B - 1)\,. \tag{8.2}$$

As a result, the only graphs which can be relevant or marginal, i.e. those which have power counting degree $\omega(G) \geq 0$, are planar two- and four-leg graphs. We will give a brief account on which role the propagator role plays in the derivation of this power counting theorem.

Four indices $\{m, n; k, \ell\} \in \mathbb{N}^2$ are associated to each internal line of a graph and two indices to each external line, thus we get $4I + 2N = 8V$ indices for a graph of genus $g = 1 - \frac{1}{2}(V - I + F)$. Since at each vertex the left index of a ribbon is identified with the right index of its neighbor, we have $4V$ independent identifications, so that we can write the indices of any propagator in terms of a set $\mathcal{I}$ of $4V$ indices. In the matrix basis the vertices are multi-dimensional Kronecker-delta functions. The amplitude of a graph G then reads

$$A_G = \sum_{\mathcal{I}} \prod_{\delta \in G} G_{m_\delta(\mathcal{I}), n_\delta(\mathcal{I}); k_\delta(\mathcal{I}), \ell_\delta(\mathcal{I})} \delta_{m_\delta - \ell_\delta, n_\delta - k_\delta}\,, \tag{8.3}$$

where the four indices of the propagator Δ of the line δ are functions of $\mathcal{I}$. Slicing of the propagator as

$$\Delta = \sum_{i=0}^{\infty} \Delta^i \quad \text{through} \quad \int_0^1 \mathrm{d}\alpha = \sum_{i=0}^{\infty} \int_{M^{-2i}}^{M^{-2(i-1)}} \mathrm{d}\alpha \tag{8.4}$$

with $M > 1$ leads to a decomposition of the amplitude as

$$\begin{aligned} A_G &= \sum_{\mu} A_{G,\mu}\,, && (8.5)\\ A_{G,\mu} &= \sum_{\mathcal{I}} \prod_{\delta \in G} \Delta^{i_\delta}_{m_\delta(\mathcal{I}), n_\delta(\mathcal{I}); k_\delta(\mathcal{I}), \ell_\delta(\mathcal{I})} \delta_{m_\delta - \ell_\delta, n_\delta - k_\delta}\,, && (8.6) \end{aligned}$$

where $\mu = \{i_\delta\}$ runs over all possible assignments of a positive integer i_δ to each line δ. The next important step is to find appropriate bounds on the propagators.

The main bounds are given by [RVTW06][1]

$$\begin{aligned} \Delta^i_{m,n;k,\ell} &\leq K\, M^{-i}\, \mathrm{e}^{-cM^{-i}(\|m\|+\|n\|+\|k\|+\|\ell\|)} && (8.7)\\ \sum_{\ell} \max_{n,k} \Delta^i_{m,n;k,\ell} &\leq K'\, M^{-i}\, \mathrm{e}^{-c' M^{-i}\|m\|} && (8.8) \end{aligned}$$

for some constants K, K' and c, c'. About half of the $4V$ indices are determined by the external indices and the Kronecker-deltas in (8.3). The undetermined indices are summation indices. Perturbative power counting amounts to finding which summations cost a factor M^{2i} through (8.7)

$$\sum_{m^1, m^2}^{\infty} \mathrm{e}^{-cM^{-i}(m^1+m^2)} = \frac{1}{(1 - \mathrm{e}^{-cM^{-i}})^2} = \frac{M^{2i}}{c^2}(1 + \mathcal{O}(M^{-i}))\,. \tag{8.9}$$

[1] For technical reasons these bounds where derived only for restricted values of Ω. This limitation has been overcome in [GMRVT06] using direct space methods.

and which cost $\mathcal{O}(1)$ due to the bound (8.8). Integrating out loops at higher scales of a graph then gives effective coupling constants in powers of M. The important point is that the faster the propagator decays, the smaller is the contribution of the integration over internal lines to effective coupling constants. This in turn reduces the number of divergent graphs. One can prove that all relevant and marginal graphs are planar four-leg and two-leg subgraphs with a single external face, which must be renormalized by counterterms. Due to symmetries there are only four initial conditions which have to be fixed by "experiments". All relevant and marginal counterterms which are needed are of Moyal-type, thus of the same form as the initial Lagrangian and can be absorbed in a redefinition of the coupling parameters Ω, g, μ and a field strength renormalization. The theory is renormalizable to all orders in perturbation theory.

8.2 Propagators

We start the renormalization program by calculating the propagators for the different models. The purpose of the first sections is to enhance the formulas given in [GRVT06] to the Minkowskian regime. In the following the propagators for the general LSZ theorem in generic $2n$ dimensions in position and in matrix basis will be given.

8.2.1 Position Space representation

The main theorem, from which all causal propagators in Minkowski space and its Euclidean counterparts can be derived, is the following theorem. The coordinates in 2n dimensions are denoted by $\boldsymbol{x} = (x^0, \dots, x^d)$ and $\boldsymbol{x}_k = (x^{2k-2}, x^{2k-1})$ with $k = 1, \dots, n$. As in section 7.1.1 we define the map $(\cdot,\cdot)_\vartheta : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{C}$ through

$$(\boldsymbol{x}, \boldsymbol{x}')_\vartheta = \cos(\vartheta)\,(\boldsymbol{x}, \boldsymbol{x}')_E + \mathrm{i}\,\sin(\vartheta)\,(\boldsymbol{x}, \boldsymbol{x}')_M\,, \tag{8.10}$$

where $(\cdot,\cdot)_M$ is the two dimensional Minkowskian and $(\cdot,\cdot)_E$ the two dimensional Euclidean scalar product. In addition we define the map $\|\cdot\| : \mathbb{R}^2 \to \mathbb{C}$ through

$$\begin{aligned} \|\boldsymbol{x}\|_\vartheta^2 &= (\boldsymbol{x}, \boldsymbol{x})_\vartheta \\ &= \cos(\vartheta)\|\boldsymbol{x}\|_E + \mathrm{i}\,\sin(\vartheta)\|\boldsymbol{x}\|_M \end{aligned} \tag{8.11}$$

with $\|\cdot\|_E$ the two dimensional Euclidean and $\|\cdot\|_M$ the two dimensional Minkowskian norm. Then we find:

Theorem 8.1. *The propagator of the regularized, general LSZ model in* 2n dimensions *is given by*

$$\begin{aligned} \Delta^{(\epsilon,\sigma)}(\boldsymbol{x}, \boldsymbol{x}') = -\,\mathrm{i}\,\mathrm{e}^{-\,\mathrm{i}\,\vartheta}\frac{E}{2\pi}\int_0^\infty \mathrm{d}s\, \frac{1}{\sinh(2sE_{-\vartheta})} \exp\left\{-\frac{\sinh(2s\tilde{E}_{-\vartheta})}{\sinh(2sE_{-\vartheta})}\,\mathrm{i}\,\boldsymbol{x}_1\cdot\boldsymbol{E}\cdot\boldsymbol{x}_1'\right\} \\ \times\quad \exp\left\{-\frac{1}{2}\coth(2sE_{-\vartheta})E(\|\boldsymbol{x}_1\|_\vartheta^2 + \|\boldsymbol{x}_1'\|_\vartheta^2) + \frac{\cosh(2s\tilde{E}_{-\vartheta})}{\sinh(2sE_{-\vartheta})}E(\boldsymbol{x}_1, \boldsymbol{x}_1')_\vartheta\right\} \\ \times\quad \prod_{k=2}^{n}\frac{B_k}{2\pi}\frac{1}{\sinh(2sB_k)}\exp\left\{-\frac{\sinh(2s\tilde{B}_k)}{\sinh(2sB_k)}\,\mathrm{i}\,\boldsymbol{x}_k\cdot\boldsymbol{B}_k\cdot\boldsymbol{x}_k'\right\} \\ \times\quad \exp\left\{-\frac{1}{2}\coth(2sB_k)B_k(\|\boldsymbol{x}_k\|_0^2 + \|\boldsymbol{x}_k'\|_0^2) + \frac{\cosh(2s\tilde{B}_k)}{\sinh(2sB_k)}B_k(\boldsymbol{x}_k, \boldsymbol{x}_k')_0\right\} \end{aligned} \tag{8.12}$$

with $\vartheta = \pi/2 - \epsilon > 0$, $\tilde{E} = (2\sigma - 1)E$ *and* $\tilde{B}_k = (2\sigma - 1)B_k$.

The proof is given in appendix I.

We can now read off the causal propagators of the relevant cases for the four dimensional LSZ and GW models. Noting that $(\cdot,\cdot)_{\pi/2} = \mathrm{i}\,(\cdot,\cdot)_M$ and thus $\|\cdot\|_{\pi/2} = \mathrm{i}\,\|\cdot\|_M$, one finds for general σ

Corollary 8.2. *The causal propagator of the general LSZ model in four-dimensional Minkowski spacetime is given by*

$$\begin{aligned}\Delta^M_{\mathsf{LSZ}}(\boldsymbol{x},\boldsymbol{x}';\sigma) &= -\frac{\mathrm{i}\,EB}{(2\pi)^2}\int_0^\infty \mathrm{d}s\,\frac{1}{\sin(2sE)}\frac{1}{\sinh(2sB)}\exp\left\{-s\mu^2 - A - B\right\}\\ &\times \quad \exp\left\{-\frac{\sin(2s\tilde{E})}{\sin(2sE)}\,\mathrm{i}\,\boldsymbol{x}_1\cdot\boldsymbol{E}\cdot\boldsymbol{x}'_1 - \frac{\sinh(2s\tilde{B})}{\sinh(2sB)}\,\mathrm{i}\,\boldsymbol{x}_2\cdot\boldsymbol{B}\cdot\boldsymbol{x}'_2\right\}\end{aligned} \tag{8.13}$$

with

$$A = -\frac{1}{2}\cot(2sE)E\left(\|\boldsymbol{x}_1\|^2_M + \|\boldsymbol{x}'_1\|^2_M\right) + \frac{\cos(2s\tilde{E})}{\sin(2sE)}E(\boldsymbol{x}_1,\boldsymbol{x}'_1)^2_M \tag{8.14}$$

and

$$B = \frac{1}{2}\coth(2sB)B\left(\|\boldsymbol{x}_2\|^2_E + \|\boldsymbol{x}'_2\|^2_E\right) - \frac{\cosh(2s\tilde{B})}{\sinh(2sB)}B(\boldsymbol{x}_2,\boldsymbol{x}'_2)^2_E\,. \tag{8.15}$$

At $\sigma = 1$ this reduces to

Corollary 8.3. *The causal propagator of the four-dimensional LSZ model for $\sigma = 1$ in Minkowski spacetime is given by*

$$\begin{aligned}\Delta^M_{\mathsf{LSZ}}(\boldsymbol{x},\boldsymbol{x}',\sigma=1) &= -\frac{\mathrm{i}\,EB}{(2\pi)^2}\,\mathrm{e}^{-\mathrm{i}\,\boldsymbol{x}_1\cdot\boldsymbol{E}\cdot\boldsymbol{x}'_1 - \mathrm{i}\,\boldsymbol{x}_2\cdot\boldsymbol{B}\cdot\boldsymbol{x}'_2}\int_0^\infty \mathrm{d}s\,\frac{1}{\sin(2sE)}\frac{1}{\sinh(2sB)}\\ &\times \quad \exp\left\{-s\mu^2 + \frac{1}{2}E\|\boldsymbol{x}_1-\boldsymbol{x}'_1\|^2_M\cot(s2E) - \frac{1}{2}B\|\boldsymbol{x}_2-\boldsymbol{x}'_2\|^2_E\coth(s2B)\right\}\end{aligned} \tag{8.16}$$

The propagator of the four dimensional Grosse-Wulkenhaar model reads:

Lemma 8.4. *The causal propagator of the Grosse-Wulkenhaar model in four-dimensional Minkowski spacetime is given by*

$$\begin{aligned}\Delta^M_{\mathsf{GW}}(\boldsymbol{x},\boldsymbol{x}') &= -\frac{\mathrm{i}\,EB}{(2\pi)^2}\int_0^\infty \mathrm{d}s\,\mathrm{e}^{-s\mu^2}\frac{1}{\sin(2sE)}\frac{1}{\sinh(2sB)}\\ &\times \quad \exp\left\{\frac{1}{2}E\cot(2sE)\left(\|\boldsymbol{x}_1\|^2_M + \|\boldsymbol{x}'_1\|^2_M\right) - \frac{E}{\sin(2sE)}(\boldsymbol{x}_1,\boldsymbol{x}'_1)_M\right\}\\ &\times \quad \exp\left\{-\frac{1}{2}B\coth(2sB)\left(\|\boldsymbol{x}_2\|^2_E + \|\boldsymbol{x}'_2\|^2_E\right) + \frac{B}{\sinh(2sB)}(\boldsymbol{x}_2,\boldsymbol{x}'_2)_E\right\}\,.\end{aligned} \tag{8.17}$$

One should notice that the Euclidean results coincide with those determined in [GRVT06]. To conform to their notation one has to substitute $\tilde{E} \to -B/2$ and $E \to \Omega/2$ within the hyperbolic functions of the LSZ model propagators.

8.2.2 Propagators in Matrix Space

Theorem 8.5. *The matrix propagator for the $2n$-dimensional regularized LSZ model in Minkowski spacetime is given by*

$$\begin{aligned}&\Delta^{(\epsilon,\sigma)}_{\boldsymbol{m},\boldsymbol{m}+\boldsymbol{\alpha};\boldsymbol{\ell}+\boldsymbol{\alpha},\boldsymbol{\ell}}\\ &= \quad -\,\mathrm{e}^{\,\mathrm{i}\,\epsilon}\frac{\theta}{8\Omega}\int_0^1 \mathrm{d}z\,z^{-\mathrm{i}\,\mathrm{e}^{\,\mathrm{i}\,\epsilon}(\sigma\alpha_1+1/2)+\sum_{i=2}^n(\sigma\alpha_i+1/2)-1+\frac{\theta\mu^2}{8\Omega}}\\ &\times \quad \Delta^{(\epsilon)}_{n_1,n_1+\alpha_1;\ell_1+\alpha_1,\ell_1}\prod_{i=2}^n\Delta^{(E)}_{n_i,n_i+\alpha_i;\ell_i+\alpha_i,\ell_i}\end{aligned} \tag{8.18}$$

with Minkowskian part

$$\Delta^{(\epsilon)}_{m,m+\alpha;\ell+\alpha,\ell} = \sum_{u=\max(0,-\alpha)}^{\min(m,\ell)} \frac{z^{-\mathrm{i}\,\mathrm{e}^{\,\mathrm{i}\,\epsilon}u}(1-z^{-i\,\mathrm{e}^{\,\mathrm{i}\,\epsilon}})^{m+\ell-2u}}{\left(1-\frac{(1-\Omega)^2}{(1+\Omega)^2}z^{-\mathrm{i}\,\mathrm{e}^{\,\mathrm{i}\,\epsilon}}\right)^{\alpha+m+\ell+1}} \left(\frac{4\Omega}{(1+\Omega)^2}\right)^{\alpha+2u+1} \left(\frac{1-\Omega}{1+\Omega}\right)^{m+\ell-2u} \mathcal{A}(m,\ell,\alpha,u) \tag{8.19}$$

and Euclidean part

$$\Delta^{(E)}_{m,m+\alpha;\ell+\alpha,\ell} = \sum_{u=\max(0,-\alpha)}^{\min(m,\ell)} \frac{z^{u}(1-z)^{m+\ell-2u}}{\left(1-\frac{(1-\Omega)^2}{(1+\Omega)^2}z\right)^{\alpha+m+\ell+1}} \left(\frac{4\Omega}{(1+\Omega)^2}\right)^{\alpha+2u+1} \left(\frac{1-\Omega}{1+\Omega}\right)^{m+\ell-2u} \mathcal{A}(m,\ell,\alpha,u)\,, \tag{8.20}$$

where

$$\mathcal{A}(n,\ell,\alpha,u) = \sqrt{\binom{\alpha+n}{\alpha+u}\binom{\alpha+\ell}{\alpha+u}\binom{n}{u}\binom{\ell}{u}}\,. \tag{8.21}$$

and $\boldsymbol{\alpha} = (\alpha_1,\ldots,\alpha_n) \in \mathbb{Z}^n$ *and* $\alpha_i = n_i - m_i$.

The proof can be found in appendix J. The respective special cases, like the four-dimensional Grosse-Wulkenhaar model etc., can easily be read off from this expression.

8.2.3 Asymptotics

We have seen that the asymptotics of the propagators play an important role for the renormalization program. But in addition we are also interested in the question whether the matrix basis makes sense at all for the description of the perturbative analysis of the LS-covariant models. Also here do the asymptotics give us the crucial information. However, the asymptotics of the Minkowskian part of the propagators are difficult to investigate due to the oscillatory behavior of its integrand. Let us consider the two-dimensional Euclidean GW operator

$$\Delta^E_{\mathsf{GW}}(\boldsymbol{x},\boldsymbol{x}') = \frac{B}{(2\pi)}\int_0^\infty \mathrm{d}s\,\frac{\mathrm{e}^{\,-s\mu^2}}{\sinh(2sB)}\exp\left\{-\frac{1}{2}B\coth(2sB)\left(x_i^2+x_i'^2\right)+\frac{B}{\sinh(2sB)}\boldsymbol{x}\cdot\boldsymbol{x}'\right\}\,. \tag{8.22}$$

Introducing *short variables* $u_i = x_i - x_i'$ and *long variables* $v_i = x_i + x_i'$ and using

$$\begin{aligned} 1 &= \cosh^2(y/2) - \sinh^2(y/2) \\ \cosh(y) &= \cosh^2(y/2) + \sinh^2(y/2) \\ \sinh(y) &= 2\sinh(y/2)\cosh(y/2)\,, \end{aligned} \tag{8.23}$$

we can rearrange

$$\begin{aligned} &-\frac{B}{2}\coth(2sB)\left(x_i^2+x_i'^2\right)+\frac{B}{\sinh(2sB)}\boldsymbol{x}\cdot\boldsymbol{x} \\ &= -\frac{B}{4}\left(\frac{\cosh^2(sB)+\sinh^2(sB)}{\cosh(sB)\sinh(sB)}\right)\left(x_i^2+x_i'^2\right)+\frac{B}{4}\left(\frac{\cosh^2(sB)-\sinh^2(sB)}{\cosh(sB)\sinh(sB)}\right)2\boldsymbol{x}\cdot\boldsymbol{x}' \\ &= -\frac{B}{4}\coth(sB)u_i^2 - \frac{B}{4}\tanh(sB)v_i^2 \end{aligned} \tag{8.24}$$

and thus

$$\Delta^E_{\mathsf{GW}}(\boldsymbol{u},\boldsymbol{v}) = \frac{1}{(2\pi)}\int_0^\infty \mathrm{d}s\,\frac{\mathrm{e}^{\,-s\frac{\mu^2}{B}}}{\sinh(2s)}\exp\left\{-\frac{B}{4}\coth(s)u_i^2 - \frac{B}{4}\tanh(s)v_i^2\right\}\,. \tag{8.25}$$

The integral is sliced in the usual way

$$\Delta^{E,i}_{\mathsf{GW}}(\boldsymbol{u},\boldsymbol{v}) = \frac{1}{(2\pi)}\int_{M^{-2i}}^{M^{-2(i-1)}} \mathrm{d}s\, \frac{\mathrm{e}^{-s\frac{\mu^2}{B}}}{\sinh(2s)} \exp\left\{-\frac{B}{4}\coth(s)u_i^2 - \frac{B}{4}\tanh(s)v_i^2\right\} \tag{8.26}$$

with $M > 1$. This can easily be estimated from above by maximizing each factor in the integrand on the interval $[M^{-2i}, M^{-2(i-1)}]$. The factor $\mathrm{e}^{-\frac{B}{4}\tanh(s)v_i^2}$ takes its maximum at $s = M^{-2i}$ at which $\tanh(s) \approx M^{-2i} - M^{-6i}/3 < c'M^{-i}$ for some constant c', while $\mathrm{e}^{-\frac{B}{4}\coth(s)u_i^2}$ takes its maximum at $s = M^{-2(i-1)}$ with $\coth(s) < M^{2(i-1)} + M^{-2(i-1)} < c''M^i$ and some constant c''. The $\sinh(2s)^{-1}$ can be estimated from above by M^{2i} such that we get the very rough bound

$$\Delta^{E,i}_{\mathsf{GW}}(\boldsymbol{u},\boldsymbol{v}) \leq KM^{2i}\,\mathrm{e}^{-c(M^i u_i^2 + M^{-i}v_i^2)} \tag{8.27}$$

for some constants K and c. This reproduces the first bound which is needed for the renormalization proof.

However, the four-dimensional Minkowski propagator in short and long variables reads

$$\begin{aligned}\Delta^{M}_{\mathsf{GW}}(\boldsymbol{x},\boldsymbol{x}') \quad &= \quad -\frac{\mathrm{i}B}{(2\pi)^2}\int_0^\infty \mathrm{d}s\, \mathrm{e}^{-s\frac{\mu^2}{B}}\frac{1}{\sin(2sB)}\frac{1}{\sinh(2sB)} \\ &\quad\times \quad \exp\left\{\frac{B}{4}\cot(s)u_{1,\mu}^2 - \frac{B}{4}\tan(s)v_{1,\mu}^2\right\} \\ &\quad\times \quad \exp\left\{-\frac{B}{4}\coth(s)u_{2,i}^2 - \frac{B}{4}\tanh(s)v_{2,i}^2\right\}\,,\end{aligned} \tag{8.28}$$

where we set $E = B$. After the slicing we can estimate the Euclidean part from above exactly as before:

$$|\Delta^{M,i}_{\mathsf{GW}}(\boldsymbol{x},\boldsymbol{x}')| \leq KM^{2i}\int_{M^{-2i}}^{M^{-2(i-1)}} \mathrm{d}s\, \frac{\mathrm{e}^{-s\frac{\mu^2}{B}}}{\sin(2sE)}\exp\left\{\frac{B}{4}\cot(s)u_{1,\mu}^2 - \frac{B}{4}\tan(s)v_{1,\mu}^2\right\}\mathrm{e}^{-c(M^i u_i^2 + M^{-i}v_i^2)}\,, \tag{8.29}$$

but the behavior of the propagator remains unclear. The Minkowskian part of the integrand is oscillating such that more sophisticated methods have to be used to estimate this integral.

There is a special case for which we can deduce the qualitative behavior. The propagator of the regularized, massless LSZ model in two dimensions for $\sigma = 1$ can be written as

$$\Delta^{(\epsilon,\sigma=1)}(\boldsymbol{x},\boldsymbol{x}') \quad = \quad -\frac{\mathrm{i}E}{2\pi}\int_0^\infty \mathrm{d}s\, \frac{1}{\sinh(2sE)}\,\mathrm{e}^{-\mathrm{i}\boldsymbol{x}\cdot\boldsymbol{E}\cdot\boldsymbol{x}'}\exp\left\{-\coth(2sE)\frac{E}{2}(\|\boldsymbol{x}-\boldsymbol{x}'\|^2_\vartheta)\right\}\,, \tag{8.30}$$

where the integration contour has been rotated as $s \to s\,\mathrm{e}^{\mathrm{i}\vartheta}$. Substituting $u = \frac{E\|\boldsymbol{x}-\boldsymbol{x}'\|^2_\vartheta}{2}(\coth(2sE) - 1)$, we get

$$\begin{aligned}&-\frac{\mathrm{i}}{4\pi}\int_0^\infty \mathrm{d}u\, \mathrm{e}^{-\mathrm{i}\boldsymbol{x}\cdot\boldsymbol{E}\cdot\boldsymbol{x}'}\frac{\mathrm{e}^{-u-\frac{E}{2}\|\boldsymbol{x}-\boldsymbol{x}'\|^2_\vartheta}}{\sqrt{u^2 + Eu\|\boldsymbol{x}-\boldsymbol{x}'\|^2_\vartheta}} \\ &= \quad -\frac{\mathrm{i}}{4\pi}\,\mathrm{e}^{-\mathrm{i}\boldsymbol{x}\cdot\boldsymbol{E}\cdot\boldsymbol{x}'}K_0\left(\frac{E}{2}\|\boldsymbol{x}-\boldsymbol{x}'\|^2_\vartheta\right)\,,\end{aligned} \tag{8.31}$$

with K_0 the modified Bessel function of the second kind of order 0. This implies that there is still a UV singularity at $\boldsymbol{x} = \boldsymbol{x}'$ due to the singular behavior of $K_0(z)$ at $z = 0$. Using the identity 9.7.2 of [AS70]

$$K_0(z) \sim \sqrt{\frac{\pi}{2z}}\,\mathrm{e}^{-z}\left(1 + \mathcal{O}(z^{-1})\right)\,, \tag{8.32}$$

we also see that $\Delta^{(\vartheta)}_{\mathsf{LSZ}}$ has an exponential decay in the *short variable* $|\boldsymbol{x}-\boldsymbol{x}'| \to \infty$ only for

$$\mathfrak{Re}\left(\|\boldsymbol{x}-\boldsymbol{x}'\|^2_\vartheta\right) > 0\,, \tag{8.33}$$

and thus only for $|\vartheta| < \pi/2$.[2] We are thus tempted to conjecture that for $\sigma < 1$ the exponential decay in $|\boldsymbol{x}+\boldsymbol{x}'| \to \infty$ also persists as long $|\vartheta| < \pi/2$. We conclude that the propagator has a worse behavior in

[2] Note that $\|\cdot\|_{\pi/2} = \mathrm{i}\|\cdot\|_M$.

Minkowski spacetime than in Euclidean spacetime, but, we can control its asymptotic behavior with help of the parameter ϑ. Considering the assumption $|\vartheta| < \pi/2$ as part of the regularization, one could try to renormalize the Minkowskian LS-covariant models.

Concerning the matrix representation we have a similar problem, since the integrand in the expression (8.18) is oscillating. Thus estimating the absolute value of the integral through an integral over the absolute value of the integrand possibly produces a big error and might lead to bad estimates on the asymptotics. Indeed, one can use this approximation to show that the Minkowskian GW propagator at $|\vartheta| = \pi/2$ has an exponential decay in each index separately. To find the other bounds, however, one has to take care of the oscillating behavior of the integrand. At least the asymptotics of the special case (8.32) for $|\vartheta| < \pi/2$ raises the hope that the propagators at hand may have such an asymptotic behavior in position space such that the matrix basis is applicable.[3]

We summarize that the questions whether the matrix representations of LS-covariant NCQFTs in terms of generalized Landau functions are well-defined and whether they are renormalizable are still open issues, but deserve a thorough investigation.

[3]Note that these propagators are LS-covariant, which implies a similar decay in momentum space.

Conclusion and Outlook

The goal of this thesis was to define LS-covariant models on Minkowski spacetime, find their renormalization properties and discuss the unitarity of the S-matrix. We briefly introduced these models on Euclidean space and showed, how the Weyl-Wigner correspondence can be used to relate their wave operators to the harmonic oscillator. Using their well-known eigenfunctions we were able to derive the eigenfunctions of the wave operators and map the Euclidean LS-covariant models onto matrix models. On Minkowski spacetime, the additional background field, which was supposed to render the models LS-covariant, spoils the vacuum persistence with respect to pair creation. Contrary to the harmonic oscillator in the Euclidean case, the Minkowskian models correspond to an inverted harmonic oscillator, implying that the wave operators do not possess a countable infinite set of eigenfunctions, which could be used to map the models onto a matrix model, but a continuously parameterized eigenbasis.

We derived the eigenfunctions of the inverted harmonic oscillator and discovered a countable infinite set of poles through an analytically continuation of these functions to the complex energy plane. The corresponding residues were identified as resonance states of the model. In order to employ an expansion of the actions in terms of these resonances we regularized the models such that the resonances turn into genuine eigenfunctions of the regularized wave operators. These operators correspond to the complex harmonic oscillator, which mediates between the ordinary to the inverted harmonic oscillator and thus between the Euclidean with the Minkowskian models, unifying both theories into one formulation related by a single parameter ϑ. We have shown that this regularized matrix basis is a bi-orthogonal system which spans the space of square-integrable functions and derived upper bounds on the asymptotics of the corresponding Hermite coefficients for tempered distributions and Gel'fand-Shilov functions. At the quantum level and in the limit of vanishing background, this regularization turned into the usual $\mathrm{i}\,\epsilon$-prescription. For the special case of a Klein-Gordon theory in a constant, external field, where the different propagators are known, we recalculated the propagator using the matrix basis and verified that the ϑ-regularization leads to Feynman propagators and thus confirmed the equivalence to the $\mathrm{i}\,\epsilon$-prescription.

We gave a short overview of the unitarity problem for models with unstable vacuum and discussed the steps which are needed to decide whether the S-matrix is unitary or not. The matrix basis was also compared to the continuous basis approach. Special divergences which are present in the continuous approach at $\Omega = 1$ are absent in the matrix representation. In turn, using the ϑ-regularization we showed that a cutoff could be employed to render the LS-covariant NCQFT finite at every step in perturbation theory and at the same time keep the LS-covariance manifestly. We derived the propagators for the regularized LS-covariant models which included the Euclidean propagators and the Minkowskian causal propagators as special cases. Due to the oscillatory behavior of the occurring integrands in Minkowski spacetime the corresponding asymptotics are much more difficult to derive than in the Euclidean case. For the special case of the massless LSZ model at $\sigma = 1$ we found that the exponential decay of the short variable in the Euclidean space vanishes if one goes to Minkowski spacetime, however persists in the near neighborhood of this case. The ϑ-regularization thus gives us a mean to control the decay behavior of the propagators. The applicability of the matrix basis in this case, however, is still in question.

We propose the following interesting perspectives for future research:

- The construction of a renormalizable and non-trivial quantum field theory in four-dimensional Minkowski spacetime is yet an unsolved problem. Encouraged by the results in Euclidean space we conclude that the LS-covariant theories in Minkowski spacetime are natural candidates and deserve a closer investigation. To probe their renormalization properties, the derivation of the exact asymptotics of the propagators is indispensable. Therefore the applicability of the matrix basis is of special interest and deserves a thorough and systematic inquiry. But, even if the matrix basis turns out to be inadequate for the investigation of these theories, techniques for the renormalization in position space are available and have already been successfully applied to LS-covariant theories in Euclidean space [GMRVT06, RVTW06, RT08]. The ϑ-regularization could then turn out to be a crucial ingredient.

- The question whether LS-covariant theories have a unitarity S-matrix has not been decided yet. Along the lines explained in section 7.4 one could try to give an answer to this question. Even if the unitarity is violated, the possibility to extend these models such as to retain unitarity is an interesting perspective, which could shed light on the construction of unitary NCQFTs in the framework of modified Feynman rules.
- The possible applications of the matrix basis are not restricted to the noncommutative LS-covariant theories. Comparing to analog calculations in a continuous eigenbasis [Rit78], calculations in the matrix basis are surprisingly simple and can thus be seen as a computational tool simplifying otherwise cumbersome calculations. It may find an application in QED and NCQED in strong external fields [Rin01, HI09, Dun09, ILM10]. The former is of fundamental theoretical interest, since the experimental observation of pair creation or other strong field phenomena would verify the validity of QED in the superstrong field domain beyond perturbation theory. There has been a resurgence of interest in these issues caused by new experiments as the "Extreme Light Infrastructure" (ELI) project [4], which will provide lasers with electromagnetic fields with unprecedented intensity, and may thus provide new insights in the non-perturbative regime of QED and QFT in general. Theoretically new techniques will be needed to realistically represent the experimental laser configurations. In this respect the effective action plays a central role, which for the constant field case has been calculated using the matrix basis in appendix F. The application to realistic experiments includes varying field configurations, which could be handled perturbatively around the constant field, which might turn out to be computational feasable with help of the ϑ-regularization and the matrix basis. For this a knowledge of the general applicability of the generalized Landau basis would be desirable.

[4] http://www.extreme-light-infrastructure.eu/

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Appendix A

Proof of Lemma 3.1

Lemma (3.1)**.** *The multiple star product of functions $f_k \in \mathcal{S}(\mathbb{R}^D)$ for $k = 1, \dots 4$ we have the following momentum and position space representations*

$$\begin{aligned}\int \mathrm{d}^D \boldsymbol{x}\ (f_1 \star f_2 \star f_3 \star f_4)(\boldsymbol{x}) &= \prod_{a=1}^{4} \left(\int \frac{\mathrm{d}^{2n} \boldsymbol{x}_a}{(2\pi)^n} \right) f(\boldsymbol{x}_1)\, f(\boldsymbol{x}_2)\, f(\boldsymbol{x}_3)\, f(\boldsymbol{x}_4)\, V(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \boldsymbol{x}_4) \\ &= \prod_{a=1}^{4} \left(\int \frac{\mathrm{d}^{2n} \boldsymbol{k}_a}{(2\pi)^n} \right) \hat{f}(\boldsymbol{k}_1)\, \hat{f}(\boldsymbol{k}_2)\, \hat{f}(\boldsymbol{k}_3)\, \hat{f}(\boldsymbol{k}_4)\, \hat{V}(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3, \boldsymbol{k}_4) \end{aligned} \tag{A.1}$$

with vertex functions

$$V(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \boldsymbol{x}_4) = \frac{(2\pi)^{2n}}{|\det(\Theta/2)|} \delta^{2n}(\boldsymbol{x}_1 - \boldsymbol{x}_2 + \boldsymbol{x}_3 - \boldsymbol{x}_4)\, \mathrm{e}^{-2\,\mathrm{i}\,(\Theta^{-1})^{ij}[(\boldsymbol{x}_1)_i(\boldsymbol{x}_2)_j + (\boldsymbol{x}_3)_i(\boldsymbol{x}_4)_j]} \tag{A.2}$$

$$\hat{V}(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3, \boldsymbol{k}_4) = (2\pi)^{2n} \delta^{2n}(\boldsymbol{k}_1 + \boldsymbol{k}_2 + \boldsymbol{k}_3 + \boldsymbol{k}_4)\, \mathrm{e}^{-\frac{\mathrm{i}}{2}\Theta^{ij}[(\boldsymbol{k}_1)_i(\boldsymbol{k}_2)_j + (\boldsymbol{k}_3)_i(\boldsymbol{k}_4)_j]}\,. \tag{A.3}$$

Proof: Using the Fourier transformation

$$\hat{f}(\boldsymbol{k}) = \int_{\mathbb{R}^D} \frac{\mathrm{d}^D \boldsymbol{x}}{(2\pi)^{D/2}}\ \mathrm{e}^{-\mathrm{i}\,\boldsymbol{k}\boldsymbol{x}} f(\boldsymbol{x}) \tag{A.4}$$

we obtain the momentum space representation

$$\begin{aligned}&\int \mathrm{d}^D \boldsymbol{x}\ (f_1 \star f_2 \star f_3 \star f_4)(\boldsymbol{x}) \\ &= \prod_{a=1}^{4} \left(\int \frac{\mathrm{d}^{2n} \boldsymbol{k}_a}{(2\pi)^{D/2}} \right) \hat{f}_1(\boldsymbol{k}_1)\, \hat{f}_2(\boldsymbol{k}_2)\, \hat{f}_3(\boldsymbol{k}_3)\, \hat{f}_4(\boldsymbol{k}_4) \int \mathrm{d}^{2n} \boldsymbol{x}\ \left(\mathrm{e}^{-\mathrm{i}\,\boldsymbol{k}_1 \cdot \boldsymbol{x}} \star \mathrm{e}^{-\mathrm{i}\,\boldsymbol{k}_2 \cdot \boldsymbol{x}} \right) \left(\mathrm{e}^{-\mathrm{i}\,\boldsymbol{k}_3 \cdot \boldsymbol{x}} \star \mathrm{e}^{-\mathrm{i}\,\boldsymbol{k}_4 \cdot \boldsymbol{x}} \right) \\ &= \prod_{a=1}^{4} \left(\int \frac{\mathrm{d}^{2n} \boldsymbol{k}_a}{(2\pi)^{D/2}} \right) \hat{f}(\boldsymbol{k}_1)\, \hat{f}(\boldsymbol{k}_2)\, \hat{f}(\boldsymbol{k}_3)\, \hat{f}(\boldsymbol{k}_4)\, \hat{V}(\boldsymbol{k}_1, \boldsymbol{k}_2 \boldsymbol{k}_3 \boldsymbol{k}_4) \end{aligned} \tag{A.5}$$

with vertex function

$$\hat{V}(\boldsymbol{k}_1, \boldsymbol{k}_2 \boldsymbol{k}_3 \boldsymbol{k}_4) = (2\pi)^{2n} \delta^{2n}(\boldsymbol{k}_1 - \boldsymbol{k}_2 + \boldsymbol{k}_3 - \boldsymbol{k}_4)\, \mathrm{e}^{\frac{\mathrm{i}}{2}\Theta^{\mu\nu}[(\boldsymbol{k}_1)_\mu(\boldsymbol{k}_2)_\nu + (\boldsymbol{k}_3)_\mu(\boldsymbol{k}_4)_\nu]}\,. \tag{A.6}$$

In position space the star-product takes the same form

$$\int \mathrm{d}^D \boldsymbol{x}\, f_1 \star f_2 \star f_3 \star f_4(\boldsymbol{x}) = \prod_{a=1}^{4} \left(\int \frac{\mathrm{d}^{2n} \boldsymbol{x}_a}{(2\pi)^n} \right) f_1(\boldsymbol{x}_1)\, f_2(\boldsymbol{x}_2)\, f_3(\boldsymbol{x}_3)\, f_4(\boldsymbol{x}_4)\, V(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \boldsymbol{x}_4)\,, \tag{A.7}$$

but with vertex function given by the inverse Fourier transform

$$V(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \boldsymbol{x}_4) = \prod_{a=1}^{4} \int \frac{\mathrm{d}^{2n} \boldsymbol{k}_a}{(2\pi)^n}\ \mathrm{e}^{\,\mathrm{i}\,(\boldsymbol{k}_1 \cdot \boldsymbol{x}_1 + \boldsymbol{k}_2 \cdot \boldsymbol{x}_2 + \boldsymbol{k}_3 \cdot \boldsymbol{x}_3 + \boldsymbol{k}_4 \cdot \boldsymbol{x}_4)} \hat{V}(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3, \boldsymbol{k}_4)\,. \tag{A.8}$$

This is just a Gaussian integral and can easily be computed. Combining the $\boldsymbol{k}_a$ and $\boldsymbol{x}_a$ for $a = 1, \dots, 4$ into 8n-component vectors

$$K = (\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3, \boldsymbol{k}_4) \quad , \quad X = (\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \boldsymbol{x}_4)\,, \tag{A.9}$$

defining the skew-symmetric $8n \times 8n$ matrix

$$A_\Theta = -\frac{1}{2}\begin{pmatrix} 0 & \mathrm{i}\,\Theta & 0 & 0 \\ -\mathrm{i}\,\Theta & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{i}\,\Theta \\ 0 & 0 & -\mathrm{i}\,\Theta & 0 \end{pmatrix} \tag{A.10}$$

and using the representation $(2\pi)^{2n}\delta^{2n}(\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3+\boldsymbol{k}_4) = \int \mathrm{d}^{2n}t\, \exp(\mathrm{i}\,K\cdot T)$ with $T=(t,t,t,t)\in\mathbb{R}^{8n}$ the integral becomes

$$V(X) = \int \mathrm{d}^{2n}t \int \frac{\mathrm{d}^{8n}K}{(2\pi)^{4n}}\, \mathrm{e}^{\,\mathrm{i}\,K(T+X)-\frac{1}{2}K\cdot A_\Theta\cdot K} = \det(\Theta/2)^{-2}\int \mathrm{d}^{2n}t\, \mathrm{e}^{\,-\frac{1}{2}(T+X)\cdot(A_\Theta)^{-1}\cdot(T+X)}\,, \tag{A.11}$$

where the relation $\det(A_\Theta)=\det(\Theta/2)^4$ has been used. Since $T\cdot(A_\Theta)^{-1}\cdot T=0$ and $T\cdot(A_\Theta)^{-1}\cdot X = X\cdot(A_\Theta)^{-1}\cdot T = -\,\mathrm{i}\,(\boldsymbol{x}_1-\boldsymbol{x}_2+\boldsymbol{x}_3-\boldsymbol{x}_4)\cdot(\Theta/2)^{-1}\cdot t$, the t-integral yields

$$\begin{aligned} V(\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_3,\boldsymbol{x}_4) &= \frac{(2\pi)^{2n}}{|\det(\Theta/2)|}\delta^{2n}(\boldsymbol{x}_1-\boldsymbol{x}_2+\boldsymbol{x}_3-\boldsymbol{x}_4)\,\mathrm{e}^{\,\frac{1}{2}X\cdot(A_\Theta)^{-1}\cdot X} \\ &= \frac{(2\pi)^{2n}}{|\det(\Theta/2)|}\delta^{2n}(\boldsymbol{x}_1-\boldsymbol{x}_2+\boldsymbol{x}_3-\boldsymbol{x}_4)\,\mathrm{e}^{\,-2\,\mathrm{i}\,(\Theta^{-1})^{\mu\nu}[(\boldsymbol{x}_1)_\mu(\boldsymbol{x}_2)_\nu+(\boldsymbol{x}_3)_\mu(\boldsymbol{x}_4)_\nu]}\,. \end{aligned} \tag{A.12}$$

which proves the lemma. □

Appendix B

Transition Matrix and its Asymptotics

In order to calculate the asymptotics of the generalized Hermite coefficients in appendix C and to show that the generalized oscillator functions and generalized Landau functions span the space of square-integrable functions in appendix D, we need to derive the transition matrix

$$h_{mn}^{(\gamma,\beta)} = \int_q f_m^{(\gamma)}(q)\, f_n^{(\beta)}(q) \tag{B.1}$$

and to find its asymptotics.

B.1 Expression for the Generalized Oscillator Functions

We start with proving a convenient representation for the generalized oscillator functions defined in equation (6.17) by

$$f_n^{(\gamma_\vartheta)}(q) = \left(\frac{\sqrt{\gamma_\vartheta}}{2^n n! \sqrt{\pi}}\right)^{1/2} \mathrm{e}^{-\frac{\gamma_\vartheta}{2}q^2} H_n(\sqrt{\gamma_\vartheta} q) \tag{B.2}$$

We will need:

Proposition B.1. *The generalized harmonic oscillator functions $f_n^{(\gamma)}(q)$ can be represented as*

$$f_n^{(\gamma)}(q) = \left(\frac{\sqrt{\gamma}}{2^n n! \sqrt{\pi}}\right)^{1/2} (2\,\mathrm{i}\,)^{\frac{n}{2}} \int_{-\infty}^{\infty} \mathrm{d}a\, (-1)^n \delta^{(n)}(a)\, \mathrm{e}^{\,\mathrm{i}\, S^{(\gamma)}(q,a)} \tag{B.3}$$

with $S^{(\gamma)}(q,a) = \frac{\mathrm{i}\,\gamma}{2}q^2 - \sqrt{2\,\mathrm{i}\,\gamma}qa + \frac{a^2}{2}$.

Proof: We will show the identity

$$\int_{-\infty}^{\infty} \mathrm{d}a\, (-1)^n \delta^{(n)}(a)\, \mathrm{e}^{\,\mathrm{i}\, S^{(\vartheta)}(q,a)} = \left(-\frac{\mathrm{i}}{2}\right)^{n/2} \mathrm{e}^{-\frac{\gamma}{2}q^2} H_n\left(\sqrt{\gamma} q\right) \tag{B.4}$$

with $S^{(\gamma)}(q,a) = \frac{\mathrm{i}\,\gamma}{2}q^2 - \sqrt{2\,\mathrm{i}\,\gamma}qa + \frac{a^2}{2}$ from which the lemma follows immediately. Defining

$$\begin{aligned} y &:= \sqrt{-\,\mathrm{i}\,/2}a - \sqrt{\gamma}q \\ \partial_a &= \sqrt{-\,\mathrm{i}\,/2}\partial_y \end{aligned} \tag{B.5}$$

we get

$$\begin{aligned} \mathrm{i}\, S^{(\gamma)}(q,a) &= \mathrm{i}\left(\frac{\mathrm{i}\,\gamma}{2}q^2 - \sqrt{2\,\mathrm{i}\,\gamma}qa + \frac{a^2}{2}\right) \\ &= -\left(\frac{\gamma}{2}q^2 + \sqrt{-2\,\mathrm{i}\,\gamma}qa - \frac{\mathrm{i}\, a^2}{2}\right) \\ &= -\left(\sqrt{-\,\mathrm{i}\,/2}a - \sqrt{\gamma}q\right)^2 + \frac{\gamma}{2}q^2 \\ &=: -y^2 + \frac{\gamma}{2}q^2\,. \end{aligned} \tag{B.6}$$

Using the definition for the Hermite polynomials

$$H_n(z) = (-1)^n \, \mathrm{e}^{\, z^2} \partial_z^n \, \mathrm{e}^{\, -z^2} \tag{B.7}$$

and noting that

$$\begin{aligned} y|_{a=0} &= -\sqrt{\gamma} q \\ H_n(-z) &= (-1)^n H_n(z) \end{aligned} \tag{B.8}$$

we get

$$\begin{aligned} &\int_{-\infty}^{\infty} \mathrm{d}a \, (-1)^n \delta^{(n)}(a) \, \mathrm{e}^{\, \mathrm{i}\, S(q,a)} \\ = \quad & \int_{-\infty}^{\infty} \mathrm{d}a \, \delta(a) \left(-\frac{\mathrm{i}}{2}\right)^{n/2} \partial_y^n \, \mathrm{e}^{\, -y^2 + \frac{\gamma}{2} q^2} \\ = \quad & \left(-\frac{\mathrm{i}}{2}\right)^{n/2} \int_{-\infty}^{\infty} \mathrm{d}a \, \delta(a) \, \mathrm{e}^{\, \frac{\gamma}{2} q^2 - y^2} (-1)^n H_n\left(y\right) \\ = \quad & \left(-\frac{\mathrm{i}}{2}\right)^{n/2} \mathrm{e}^{\, -\frac{\gamma}{2} q^2} H_n\left(\sqrt{\gamma} q\right) \end{aligned} \tag{B.9}$$

which proves the lemma. □

B.2 Expression for the Transition Matrix

To switch between two sets of generalized oscillator functions $(f_n^{(\beta)})_{n\in\mathbb{N}}$ and $(f_n^{(\gamma)})_{n\in\mathbb{N}}$ we need the *transition matrix*, whose explicit form will be derived in the next proposition:

Proposition B.2. *Let $\beta, \gamma \in \mathbb{C} - \{0\}$ be two different complex numbers with $\mathfrak{Re}(\beta + \gamma) > 0$. The transition matrix*

$$h_{nm}^{(\gamma,\beta)} := \int_{-\infty}^{\infty} \mathrm{d}x \, f_n^{(\gamma)}(x) f_m^{(\beta)}(x) \tag{B.10}$$

is given by

$$\begin{aligned} h_{nm}^{(\gamma,\beta)} \quad = \quad & \left(\frac{\sqrt{\gamma}}{2^n n! \sqrt{\pi}}\right)^{1/2} \left(\frac{\sqrt{\beta}}{2^m m! \sqrt{\pi}}\right)^{1/2} \sqrt{\frac{2\pi}{\gamma+\beta}} \sqrt{\left(\frac{\gamma-\beta}{\gamma+\beta}\right)}^{m+n} \\ \times \quad & \sum_{k=\max\left(0, \frac{m-n}{2}\right)}^{\lfloor m/2 \rfloor} \frac{n! m! (-1)^k}{k! \, (m-2k)! \, (k + \frac{n-m}{2})!} \sqrt{\frac{16\beta\gamma}{(\gamma-\beta)^2}}^{m-2k} \\ \times \quad & \begin{cases} 1\,, & \frac{|m-n|}{2} \in \mathbb{N} \\ 0\,, & \frac{|m-n|}{2} \in \mathbb{N} + \frac{1}{2} \end{cases} \quad . \end{aligned} \tag{B.11}$$

Proof: Using proposition B.1 we have

$$\begin{aligned} h_{nm}^{(\gamma,\beta)} \quad = \quad & \left(\frac{\sqrt{\gamma}}{2^n n! \sqrt{\pi}}\right)^{1/2} \left(\frac{\sqrt{\beta}}{2^m m! \sqrt{\pi}}\right)^{1/2} (2\,\mathrm{i}\,)^{\frac{n+m}{2}} \\ \times \quad & \int_x \int_a \int_{a'} (-1)^n \delta^{(n)}(a) (-1)^m \delta^{(m)}(a') \, \mathrm{e}^{\, \mathrm{i}\, S^{(\gamma)}(x,a) + \mathrm{i}\, S^{(\beta)}(x,a')} \end{aligned} \tag{B.12}$$

with $S^{(\gamma)}(x,a) = \frac{\mathrm{i}\,\gamma}{2} x^2 - \sqrt{2\,\mathrm{i}\,\gamma} x a + \frac{a^2}{2}$. The exponential can then be rearranged to

$$
\begin{aligned}
&\mathrm{i}\,S^{(\gamma)}(x,a) + \mathrm{i}\,S^{(\beta)}(x,a') \\
&= -\frac{\gamma}{2}x^2 - \frac{\beta}{2}x^2 - \mathrm{i}\sqrt{2\,\mathrm{i}\,\gamma}x\,a - \mathrm{i}\sqrt{2\,\mathrm{i}\,\beta}x\,a' + \frac{\mathrm{i}}{2}a^2 + \frac{\mathrm{i}}{2}a'^2 \\
&= \underbrace{-\frac{1}{2}(\gamma+\beta)\left(x + \mathrm{i}\,\frac{\sqrt{2\,\mathrm{i}\,\gamma}\,a + \sqrt{2\,\mathrm{i}\,\beta}\,a'}{\gamma+\beta}\right)^2}_{-b\tilde{x}^2} + \frac{\mathrm{i}}{2}\frac{(a^2+a'^2)(\gamma+\beta) - 2(\sqrt{\gamma}a + \sqrt{\beta}a')^2}{\gamma+\beta} \\
&= -b\tilde{x}^2 \underbrace{-\frac{\mathrm{i}}{2}\left(\frac{\gamma-\beta}{\gamma+\beta}\right)\left[a + \frac{2\sqrt{\gamma\beta}\,a'}{\gamma-\beta}\right]^2}_{=:-y^2} + \frac{\mathrm{i}}{2}\left(\frac{\gamma-\beta}{\gamma+\beta}\right)\left[\frac{4\gamma\beta}{(\gamma-\beta)^2} + 1\right]a'^2 \\
&= -b\tilde{x}^2 - y^2 + \frac{\mathrm{i}}{2}\left(\frac{\gamma+\beta}{\gamma-\beta}\right)a'^2\,. \qquad \text{(B.13)}
\end{aligned}
$$

Since we assumed $\mathfrak{Re}(\gamma+\beta) > 0$ for $i = 1,2$ we can perform the $\tilde{x}$-integration giving

$$
\int \mathrm{d}x\;\mathrm{e}^{-b\,\tilde{x}^2} = \sqrt{\frac{2\pi}{\gamma+\beta}}\,. \qquad \text{(B.14)}
$$

With the definition

$$
y^2 \;:=\; \frac{\mathrm{i}}{2}\left(\frac{\gamma-\beta}{\gamma+\beta}\right)\left[a + \frac{2\sqrt{\gamma\beta}\,a'}{\gamma-\beta}\right]^2 \qquad \text{(B.15)}
$$

one gets

$$
\begin{aligned}
\partial_a &= \sqrt{\frac{\mathrm{i}}{2}\left(\frac{\gamma-\beta}{\gamma+\beta}\right)}\partial_y \\
y^2|_{a=0} &= \frac{\mathrm{i}}{2}\left(\frac{\gamma-\beta}{\gamma+\beta}\right)\left[\frac{2\sqrt{\gamma\beta}\,a'}{\gamma-\beta}\right]^2 = \frac{2\,\mathrm{i}\,\gamma\beta}{\gamma^2-\beta^2}a'^2\,.
\end{aligned} \qquad \text{(B.16)}
$$

We see that the a-integration leads to another Hermite polynomial

$$
\begin{aligned}
&\int_a \delta(a)\partial_a^n \exp\left\{-\frac{\mathrm{i}}{2}\left(\frac{\gamma-\beta}{\gamma+\beta}\right)\left[a - \frac{2\sqrt{\gamma\beta}\,a'}{\gamma-\beta}\right]^2 + \frac{\mathrm{i}}{2}\left(\frac{\gamma+\beta}{\gamma-\beta}\right)a'^2\right\} \\
&= \sqrt{\frac{\mathrm{i}}{2}\left(\frac{\gamma-\beta}{\gamma+\beta}\right)}^{\,n} \exp\left\{\frac{\mathrm{i}}{2}\left(\frac{\gamma+\beta}{\gamma-\beta}\right)a'^2\right\}\mathrm{e}^{-y^2}\,\mathrm{e}^{y^2}\partial_y^n\,\mathrm{e}^{-y^2}\Big|_{a=0} \\
&= \sqrt{\frac{\mathrm{i}}{2}\left(\frac{\gamma-\beta}{\gamma+\beta}\right)}^{\,n} \exp\left\{\frac{\mathrm{i}}{2}\left(\frac{\gamma+\beta}{\gamma-\beta}\right)a'^2 - \frac{2\,\mathrm{i}\,\gamma\beta}{\gamma^2-\beta^2}a'^2\right\}(-1)^n H_n\left(\sqrt{\frac{2\,\mathrm{i}\,\gamma\beta}{\gamma^2-\beta^2}}a'\right) \\
&= \sqrt{\frac{\mathrm{i}}{2}\left(\frac{\gamma-\beta}{\gamma+\beta}\right)}^{\,n} \exp\left\{\frac{\mathrm{i}}{2}\left(\frac{\gamma-\beta}{\gamma+\beta}\right)a'^2\right\}H_n\left(-\sqrt{\frac{2\,\mathrm{i}\,\gamma\beta}{\gamma^2-\beta^2}}a'\right)\,. \qquad \text{(B.17)}
\end{aligned}
$$

and our intermediate result is

$$
(2\,\mathrm{i})^{\frac{m+n}{2}}\sqrt{\frac{2\pi}{\gamma+\beta}}\sqrt{\frac{\mathrm{i}}{2}\left(\frac{\gamma-\beta}{\gamma+\beta}\right)}^{\,n}\int_{a'}\delta(a')\partial_{a'}^m\left[\exp\left\{\frac{\mathrm{i}}{2}\left(\frac{\gamma-\beta}{\gamma+\beta}\right)a'^2\right\}H_n\left(-\sqrt{\frac{2\,\mathrm{i}\,\gamma\beta}{\gamma^2-\beta^2}}a'\right)\right]\,. \qquad \text{(B.18)}
$$

In the following we will perform the derivatives

$$
\begin{aligned}
&\partial_{a'}^m\left[\mathrm{e}^{\,Aa'^2}H_n(Ba')\right]_{a'=0} \\
&= \sum_{k=0}^{m}\binom{m}{k}\left(\mathrm{e}^{\,Aa'^2}\right)^{(k)}\Big|_{a'=0} H_n^{(m-k)}(Ba')\Big|_{a'=0}\,. \qquad \text{(B.19)}
\end{aligned}
$$

Using the explicit formula for the Hermite polynomials [AS70]

$$H_n(z) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k n!}{k!(n-2k)!}(2z)^{n-2k}\,. \tag{B.20}$$

one can derive the derivatives of the respective factors:

$$\left(\mathrm{e}^{Aa'^2}\right)^{(k)}\bigg|_{a'=0} = A^{k/2}\frac{k!}{(k/2)!}\left\{\begin{array}{lll} 1\,, & \text{for} & k \quad even \\ 0\,, & \text{for} & k \quad odd \end{array}\right\} \tag{B.21}$$

and

$$\begin{aligned}
&\partial_{a'}^{m-k}\left(\sum_{\ell=0}^{\lfloor n/2\rfloor}\frac{n!(-1)^\ell}{\ell!(n-2\ell)!}(2Ba')^{n-2\ell}\right)\Bigg|_{a'=0} \\
&= (2B)^{m-k}\left(\sum_{\ell=0}^{\lfloor n/2\rfloor}\frac{n!(-1)^\ell}{\ell!(n-2\ell-m+k)!}(2Ba')^{n-2\ell-m+k}\right)\Bigg|_{a'=0}\left\{\begin{array}{lll} 1\,, & \text{for} & n \geq m-k \\ 0\,, & \text{for} & n < m-k \end{array}\right\} \\
&= (2B)^{m-k}(-1)^{\frac{n-m+k}{2}}\frac{n!}{\left(\frac{n-m+k}{2}\right)!}\left\{\begin{array}{lllr} 1\,, & \text{for} & n-m+k \quad even & \text{and } n \geq m-k \\ 0\,, & \text{for} & n-m+k \quad odd & \text{or } n < m-k \end{array}\right\}.
\end{aligned} \tag{B.22}$$

Taking care of the three conditions for non-vanishing derivatives, k even, $m \geq n-k$ and $m-n+k$ even, we get

$$\begin{aligned}
&\partial_{a'}^{m}\left[\mathrm{e}^{Aa'^2}H_n(Ba')\right]_{a'=0} \\
&= \sum_{k=0}^{m}\binom{m}{k}\left(\mathrm{e}^{Aa'^2}\right)^{(k)}\bigg|_{a'=0} H_n^{(m-k)}(Ba')\Big|_{a'=0} \\
&= \sum_{k=0}^{\lfloor m/2\rfloor}\binom{m}{2k}\frac{(-1)^{\frac{n-m+2k}{2}}(2k)!n!}{k!\left(\frac{n-m+2k}{2}\right)!}A^k(2B)^{m-2k}\left\{\begin{array}{lllr} 1\,, & \text{for} & n-m+2k \quad even & \text{and } n \geq m-2k \\ 0\,, & \text{for} & n-m+2k \quad odd & \text{or } n < m-2k \end{array}\right\} \\
&= \sum_{k=\max\left(0,\frac{m-n}{2}\right)}^{\lfloor m/2\rfloor}\frac{(-1)^{\frac{n}{2}}(-1)^{-\frac{m}{2}}(2B)^m n!m!}{k!\,(m-2k)!\,\left(k+\frac{n-m}{2}\right)!}\left(-\frac{A}{4B^2}\right)^k\left\{\begin{array}{ll} 1\,, & n-m \text{ even} \\ 0\,, & n-m \text{ odd} \end{array}\right\}.
\end{aligned} \tag{B.23}$$

Putting this into (B.18) with

$$\begin{aligned}
A &= \frac{\mathrm{i}}{2}\left(\frac{\gamma-\beta}{\gamma+\beta}\right) \\
2B &= -\sqrt{\frac{8\,\mathrm{i}\,\gamma\beta}{\gamma^2-\beta^2}} \\
-\frac{A}{4B^2} &= -\frac{\mathrm{i}}{2}\left(\frac{\gamma-\beta}{\gamma+\beta}\right)\left(\frac{\gamma^2-\beta^2}{8\,\mathrm{i}\,\gamma\beta}\right) = -\frac{\mathrm{i}}{2}\left(\frac{(\gamma-\beta)^2}{8\,\mathrm{i}\,\gamma\beta}\right) = -\left(\frac{\gamma-\beta}{4\sqrt{\gamma\beta}}\right)^2
\end{aligned} \tag{B.24}$$

and assuming "even $n-m$" we get

$$\begin{aligned}
&(2\,\mathrm{i})^{\frac{m+n}{2}}\sqrt{\frac{2\pi}{\gamma+\beta}}\sqrt{\frac{\mathrm{i}}{2}\left(\frac{\gamma-\beta}{\gamma+\beta}\right)}^{\,n}\sqrt{\frac{8\,\mathrm{i}\,\gamma\beta}{\gamma^2-\beta^2}}^{\,m}\sum_{k=\max\left(0,\frac{m-n}{2}\right)}^{\lfloor m/2\rfloor}\frac{\mathrm{i}^{\,n}n!\,\mathrm{i}^{\,m}m!}{k!\,(m-2k)!\,\left(\frac{n-m+2k}{2}\right)!}(-1)^k\left(\frac{\gamma-\beta}{4\sqrt{\gamma\beta}}\right)^{2k} \\
&= \sqrt{\frac{2\pi}{\gamma+\beta}}\sqrt{\left(\frac{\gamma-\beta}{\gamma+\beta}\right)}^{\,m+n}\sum_{k=\max\left(0,\frac{m-n}{2}\right)}^{\lfloor m/2\rfloor}\frac{n!m!(-1)^k}{k!\,(m-2k)!\,\left(k+\frac{n-m}{2}\right)!}\sqrt{\frac{16\gamma\beta}{(\gamma-\beta)^2}}^{\,m-2k}.
\end{aligned} \tag{B.25}$$

Putting this into (B.12) one finds

$$
\begin{aligned}
h_{nm}^{(\gamma,\beta)} &= \left(\frac{\sqrt{\gamma}}{2^n n!\sqrt{\pi}}\right)^{1/2}\left(\frac{\sqrt{\beta}}{2^m m!\sqrt{\pi}}\right)^{1/2}\sqrt{\frac{2\pi}{\gamma+\beta}}\sqrt{\left(\frac{\gamma-\beta}{\gamma+\beta}\right)}^{m+n} \\
&\times \sum_{k=\max\left(0,\frac{m-n}{2}\right)}^{\lfloor m/2\rfloor}\frac{n!m!(-1)^k}{k!\,(m-2k)!\,\left(k+\frac{n-m}{2}\right)!}\sqrt{\frac{16\beta\gamma}{(\gamma-\beta)^2}}^{m-2k}
\end{aligned}
\tag{B.26}
$$

for "$m-n$ even" and 0 otherwise. □

B.3 Asymptotics of the Transition Matrix

We need the asymptotic behavior of the transition matrix to determine estimations on the asymptotics of the generalized Hermite coefficients. To prove the asymptotics we bring (B.11) into a more compact form.

Lemma B.3. *The generalized transition matrix $h_{nm}^{(\gamma,\beta)}$ with $\beta,\gamma\in\mathbb{C}-\{0\}$ and $\mathfrak{Re}(\beta+\gamma)>0$ can be brought into the form*

$$
h_{nm}^{(\gamma,\beta)} = \left(\frac{4\beta\gamma}{(\beta+\gamma)^2}\right)^{1/4}\sqrt{\frac{m!}{n!}}P_{\frac{n+m}{2}}^{\frac{n-m}{2}}\left[\sqrt{\frac{4\beta\gamma}{(\gamma+\beta)^2}}\right]\begin{cases}1\,, & \frac{|m-n|}{2}\in\mathbb{N}\\ 0\,, & \frac{|m-n|}{2}\in\mathbb{N}+\frac{1}{2}\end{cases}
\tag{B.27}
$$

where P is the Legendre function of first kind.

Proof: We start with

$$
\begin{aligned}
h_{nm}^{(\gamma,\beta)} &= \left(\frac{\sqrt{\gamma}}{2^n n!\sqrt{\pi}}\right)^{1/2}\left(\frac{\sqrt{\beta}}{2^m m!\sqrt{\pi}}\right)^{1/2}\sqrt{\frac{2\pi}{\beta+\gamma}}\sqrt{\frac{\gamma-\beta}{\beta+\gamma}}^{m+n} \\
&\times \sum_{k=\max\left(0,\frac{m-n}{2}\right)}^{\lfloor m/2\rfloor}\frac{n!m!(-1)^k}{k!\,(m-2k)!\,\left(k+\frac{n-m}{2}\right)!}\sqrt{\frac{16\beta\gamma}{(\beta-\gamma)^2}}^{m-2k}\,.
\end{aligned}
\tag{B.28}
$$

It is not self-evident, but this result is invariant under exchange of $(m,\beta)\leftrightarrow(n,\gamma)$, as it should be. The following table shows, which values the different factors can depending on k and on whether we have $m<n$ or $m\geq n$:

	$m<n$	$m\geq n$
k	$0,1,\ldots,\lfloor m/2\rfloor$	$\frac{m-n}{2},\frac{m-n}{2}+1,\ldots,\lfloor m/2\rfloor$
$m-2k$	$m,m-2,\ldots,m-\lfloor m\rfloor$	$n,n-2,\ldots,n-\ldots,m-\lfloor m\rfloor$
$k+\frac{n-m}{2}$	$\frac{n-m}{2},\frac{n-m}{2}+1,\ldots,\lfloor n/2\rfloor$	$0,1,\ldots,\lfloor n/2\rfloor$

Thus in terms of a new variable $\bar{k}$ defined by

$$
\bar{k} = k+\frac{n-m}{2}
\tag{B.29}
$$

the sum gets an extra factor $(-1)^{\frac{m-n}{2}}$:

$$
\begin{aligned}
&\sum_{k=\max\left(0,\frac{m-n}{2}\right)}^{\lfloor m/2\rfloor}\frac{n!m!(-1)^k}{k!\,(m-2k)!\,\left(k+\frac{n-m}{2}\right)!}\sqrt{\frac{16\beta\gamma}{(\beta-\gamma)^2}}^{m-2k} \\
&= \sum_{\bar{k}=\max\left(0,\frac{n-m}{2}\right)}^{\lfloor n/2\rfloor}\frac{n!m!(-1)^{\frac{m-n}{2}}(-1)^{\bar{k}}}{\bar{k}!\,(n-2\bar{k})!\,\left(\bar{k}+\frac{m-n}{2}\right)!}\sqrt{\frac{16\beta\gamma}{(\beta-\gamma)^2}}^{n-2\bar{k}}\,.
\end{aligned}
\tag{B.30}
$$

The exchange of β and γ has the converse effect. The two factors depending on $\beta-\gamma$ earn extra factors

$$\begin{aligned}&\sqrt{-\left(\frac{\beta-\gamma}{\beta+\gamma}\right)}^{m+n}\sqrt{\frac{16\beta\gamma}{(\beta-\gamma)^2}}^{m-2\bar{k}}\\ &= (-1)^{\frac{n-m}{2}}\sqrt{-\left(\frac{\beta-\gamma}{\beta+\gamma}\right)}^{m+n}\sqrt{\frac{16\beta\gamma}{(\beta-\gamma)^2}}^{m-2\bar{k}}\end{aligned}\tag{B.31}$$

which cancel exactly the contribution of the redefinition in terms of $\bar{k}$. We can use equation (B.30) to rewrite the sum as

$$\begin{aligned}&\sum_{k=\max\left(0,\frac{m-n}{2}\right)}^{\lfloor m/2\rfloor}\frac{n!m!(-1)^{-\frac{m}{2}}}{k!\,(m-2k)!\,\left(k+\frac{n-m}{2}\right)!}\sqrt{\frac{16\beta\gamma}{(\mathrm{i}\,\gamma-\mathrm{i}\,\beta)^2}}^{m-2k}\\ &= \sum_{k=0}^{\lfloor\min(m,n)/2\rfloor}\frac{n!m!(-1)^{-\frac{\min(m,n)}{2}}}{k!\,(\min(m,n)-2k)!\,\left(k+\left|\frac{n-m}{2}\right|\right)!}\sqrt{\frac{16\beta\gamma}{(\mathrm{i}\,\gamma-\mathrm{i}\,\beta)^2}}^{m-2k}.\end{aligned}\tag{B.32}$$

Using

$$\sum_{k=0}^{\lfloor p/2\rfloor}\frac{p!}{k!\,(p-2k)!\,(\ell+k)!}C^k = \frac{1}{\ell!}\,{}_2F_1\left(-\frac{p}{2},\frac{1-p}{2},1+\ell,4C\right)\tag{B.33}$$

one gets

$$\begin{aligned}h^{(\gamma,\beta)}_{n,m} &= \left(\frac{\sqrt{\gamma}}{2^n n!\sqrt{\pi}}\right)^{1/2}\left(\frac{\sqrt{\beta}}{2^m m!\sqrt{\pi}}\right)^{1/2}\sqrt{\frac{2\pi}{\beta+\gamma}}\sqrt{\frac{\gamma-\beta}{\beta+\gamma}}^{m+n}\sqrt{\frac{16\beta\gamma}{(\mathrm{i}\,\gamma-\mathrm{i}\,\beta)^2}}^{\min(m,n)}\\ &\times\frac{(-1)^{-\frac{\min(m,n)}{2}}\max(m!,n!)}{\left|\frac{n-m}{2}\right|!}\,{}_2F_1\left(-\frac{\min(m,n)}{2},\frac{1-\min(m,n)}{2},1+\left|\frac{n-m}{2}\right|,\frac{(\mathrm{i}\,\gamma-\mathrm{i}\,\beta)^2}{4\beta\gamma}\right)\end{aligned}\tag{B.34}$$

which is totally symmetric in m and n.

Now we want to get rid of the minimum and maximum functions by relating the hypergeometric functions to Legendre functions P^μ_ν. The following formulas will be of use for us:

$$\begin{aligned}P^\mu_{-\nu-1}(z) &= P^\mu_\nu(z)\\ P^{-\mu}_\nu &= (-1)^\mu\frac{(\nu-\mu)!}{(\nu+\mu)!}\left[P^\mu_\nu(z)-\frac{2\pi}{\mathrm{e}}^{-\mathrm{i}\,\mu\pi}\sin(\mu\pi)Q^\mu_\nu(z)\right]\end{aligned}\tag{B.35}$$

with Q^μ_ν Legendre functions of the second kind. However, since in the following we will have "$\mu=\frac{m-n}{2}$ even" the second equation will simplify to

$$P^{-\mu}_\nu = (-1)^\mu\frac{(\nu-\mu)!}{(\nu+\mu)!}P^\mu_\nu(z)\,.\tag{B.36}$$

The important identity relating our hypergeometric function to a Legendre function is (15.4.10) of [AS70][1]

$${}_2F_1(a,a+\frac{1}{2};c;z) = 2^{c-1}\Gamma(c)(-z)^{\frac{1}{2}-\frac{1}{2}c}(1-z)^{\frac{1}{2}c-a-\frac{1}{2}}P^{1-c}_{2a-c}\left[(1-z)^{-1/2}\right]\,.\tag{B.37}$$

The factors of (B.34) depending on minimum and maximum functions are

$$2^{\min(m,n)}(-z)^{-\frac{\min(m,n)}{2}}\frac{\max(m!,n!)}{(\left|\frac{m-n}{2}\right|)!}{}_2F_1\left(\frac{1-\min(m,n)}{2},-\frac{\min(m,n)}{2},1+\left|\frac{m-n}{2}\right|,z\right)\,.\tag{B.38}$$

[1] In [AS70] is a factor $(-1)^{\frac{1}{2}-\frac{1}{2}c}$ missing which I included. This can also be seen from equation (15.4.11) in [AS70], which is exactly the same formula for real z, where this factor is present.

with

$$z = \frac{(\mathrm{i}\,\gamma - \mathrm{i}\,\beta)^2}{4\beta\gamma}\,. \tag{B.39}$$

For $m < n$ we identify

$$\begin{aligned} a = -\frac{m}{2} \quad &, \quad c = 1 + \frac{n-m}{2} \\ \frac{c-1}{2} - a = \frac{n+m}{4} \quad &, \quad 2a - c = -1 - \frac{n}{2} - \frac{m}{2} \end{aligned} \tag{B.40}$$

and use equation (B.37) to get

$$\begin{aligned} & 2^m (-z)^{-\frac{m}{2}} \frac{n!}{\left(\frac{n-m}{2}\right)!} {}_2F_1\left(-\frac{m}{2}, \frac{1-m}{2}, 1 + \frac{n-m}{2}, z\right) \\ = \quad & 2^m (-z)^{-\frac{m}{2}} \frac{n!}{\left(\frac{n-m}{2}\right)!} 2^{\frac{n-m}{2}} \left(\frac{n-m}{2}\right)! (-z)^{\frac{m-n}{4}} (1-z)^{\frac{n+m}{4}} P^{\frac{m-n}{2}}_{-1-\frac{n}{2}-\frac{m}{2}} \left[(1-z)^{-1/2}\right] \\ = \quad & n! 2^{\frac{n+m}{2}} \left(\frac{z-1}{z}\right)^{\frac{m+n}{4}} P^{\frac{m-n}{2}}_{\frac{n+m}{2}} \left[(1-z)^{-1/2}\right]\,. \end{aligned} \tag{B.41}$$

For $m \geq n$ we have

$$\begin{aligned} a = -\frac{n}{2} \quad &, \quad c = 1 + \frac{m-n}{2} \\ \frac{c-1}{2} - a = \frac{n+m}{4} \quad &, \quad 2a - c = -1 - \frac{n}{2} - \frac{m}{2} \end{aligned} \tag{B.42}$$

and thus

$$\begin{aligned} & 2^n (-z)^{-\frac{n}{2}} \frac{m!}{\left(\frac{m-n}{2}\right)!} {}_2F_1\left(-\frac{n}{2}, \frac{1-n}{2}, 1 + \frac{m-n}{2}, z\right) \\ = \quad & 2^n (-z)^{-\frac{n}{2}} \frac{m!}{\left(\frac{m-n}{2}\right)!} 2^{\frac{m-n}{2}} \left(\frac{m-n}{2}\right)! (-z)^{\frac{n-m}{4}} (1-z)^{\frac{n+m}{4}} P^{\frac{n-m}{2}}_{-1-\frac{n}{2}-\frac{m}{2}} \left[(1-z)^{-1/2}\right] \\ = \quad & m! 2^{\frac{n+m}{2}} \left(\frac{z-1}{z}\right)^{\frac{m+n}{4}} P^{\frac{n-m}{2}}_{\frac{n+m}{2}} \left[(1-z)^{-1/2}\right]\,. \end{aligned} \tag{B.43}$$

Now we can use equation (B.36) with

$$\begin{aligned} -\mu = \frac{m-n}{2} \quad &, \quad \nu = \frac{m+n}{2} \\ (\nu - \mu)! = m! \quad &, \quad (\nu + \mu)! = n! \end{aligned} \tag{B.44}$$

to see that that the expressions in terms of Legendre function coincide. Inserting the explicit expression for z and

$$\begin{aligned} \frac{z-1}{z} &= \frac{\frac{(\mathrm{i}\,\gamma - \mathrm{i}\,\beta)^2}{4\beta\gamma} - 1}{\frac{(\mathrm{i}\,\gamma - \mathrm{i}\,\beta)^2}{4\beta\gamma}} = \left(\frac{\gamma+\beta}{\gamma-\beta}\right)^2 \\ 1 - z &= \frac{(\gamma+\beta)^2}{4\beta\gamma} \end{aligned} \tag{B.45}$$

we get

$$\begin{aligned} & (-1)^{-\frac{\min(m,n)}{2}} \sqrt{\frac{16\beta\gamma}{(\mathrm{i}\,\gamma - \mathrm{i}\,\beta)^2}}^{-\frac{\min(m,n)}{2}} \frac{\max(m!, n!)}{\left(\left|\frac{m-n}{2}\right|\right)!} \\ \times \quad & {}_2F_1\left(\frac{1-\min(m,n)}{2}, -\frac{\min(m,n)}{2}, 1 + \left|\frac{m-n}{2}\right|, \frac{(\mathrm{i}\,\gamma - \mathrm{i}\,\beta)^2}{4\beta\gamma}\right) \\ = \quad & m! 2^{\frac{n+m}{2}} \sqrt{\frac{\gamma+\beta}{\gamma-\beta}}^{m+n} P^{\frac{n-m}{2}}_{\frac{n+m}{2}} \left[\sqrt{\frac{4\beta\gamma}{(\gamma+\beta)^2}}\right] \end{aligned} \tag{B.46}$$

This simplifies the matrix $h_{nm}^{(\gamma,\beta)}$:

$$\begin{aligned} h_{nm}^{(\gamma,\beta)} &= \left(\frac{\sqrt{\gamma}}{2^n n!\sqrt{\pi}}\right)^{1/2}\left(\frac{\sqrt{\beta}}{2^m m!\sqrt{\pi}}\right)^{1/2}\sqrt{\frac{2\pi}{\beta+\gamma}}\sqrt{\frac{\gamma-\beta}{\beta+\gamma}}^{m+n} m!2^{\frac{n+m}{2}}\sqrt{\frac{\gamma+\beta}{\gamma-\beta}}^{m+n} P_{\frac{n+m}{2}}^{\frac{n-m}{2}}\left[\sqrt{\frac{4\beta\gamma}{(\gamma+\beta)^2}}\right] \\ &= \left(\frac{4\beta\gamma}{(\beta+\gamma)^2}\right)^{1/4}\sqrt{\frac{m!}{n!}}P_{\frac{n+m}{2}}^{\frac{n-m}{2}}\left[\sqrt{\frac{4\beta\gamma}{(\gamma+\beta)^2}}\right] \end{aligned} \tag{B.47}$$

and proves the lemma. □

It is easy to see that the argument of the transition matrix takes values between 1 and $\sqrt{2}$, with its minimum at $\beta=\gamma$ and its maximum at $|\gamma|=|\beta|$ and $\gamma/\beta=\pm\mathrm{i}$. Below $\sqrt{2}$, the transition matrix decays exponentially in each index, as will be shown in the next lemma:

Lemma B.4. *Let $\beta,\gamma\in\mathbb{C}-\{0\}$. Then for large n the transition matrix behaves as*

$$h_{nm}^{(\gamma,\beta)} \sim n^{-1/2}\left|\frac{\beta-\gamma}{\beta+\gamma}\right|^n . \tag{B.48}$$

Proof: The Legendre function has the following integral representation (equation 14.12.8 of [OLBC10])

$$\begin{aligned} P_\nu^\mu(x) &= \frac{2^\mu\mu!(\nu+\mu)!(x^2-1)^\mu}{(2\mu)!(\nu-\mu)!\pi}\int_0^\pi \mathrm{d}\phi\,\left(x+\sqrt{x^2-1}\cos\phi\right)^{\nu-\mu}(\sin\phi)^{2\mu} \\ &= \frac{2^{\frac{n-m}{2}}\left(\frac{n-m}{2}\right)!n!(x^2-1)^{\frac{n-m}{2}}}{(n-m)!m!\pi}\int_0^\pi \mathrm{d}\phi\,\left(x+\sqrt{x^2-1}\cos\phi\right)^m(\sin\phi)^{n-m}, \end{aligned} \tag{B.49}$$

where in our case $2\mu=n-m$ and $2\nu=n+m$ and

$$x := \sqrt{\frac{4\beta\gamma}{(\beta+\gamma)^2}}\,. \tag{B.50}$$

The integral can be estimated via the saddle point method for large n. Writing the integral in the form

$$\begin{aligned} &\int_0^\pi \mathrm{d}\phi\,\left(x+\sqrt{x^2-1}\cos\phi\right)^m(\sin\phi)^{n-m} \\ &= \int_0^\pi \mathrm{d}\phi\;\mathrm{e}^{n\ln(\sin\phi)}\left(\frac{x+\sqrt{x^2-1}\cos\phi}{\sin\phi}\right)^m \\ &= \int_0^{\pi/2}\mathrm{d}\phi\;\mathrm{e}^{n\omega(\phi)}\varphi(\phi)\,. \end{aligned} \tag{B.51}$$

the exponential has one saddle point at $\phi=\pi/2$. Expanding around $\pi/2$ gives $\omega(\phi)=1-\frac{1}{2!}(\sin(\pi/2))^{-2}(\phi-\pi/2)^2+\ldots=1-\frac{1}{2}(\phi-\pi/2)^2+\ldots$, which gives the saddle point approximation [Cop65]

$$\begin{aligned} \int_0^{\pi/2}\mathrm{d}\phi\;\mathrm{e}^{n\omega(\phi)}\varphi(\phi) &= \varphi(\pi/2)\,\mathrm{e}^{n\omega(\pi/2)}\left(\frac{-2\pi}{n\omega''(\pi/2)}\right)^{1/2}\left(1+\mathcal{O}(n^{-\epsilon})\right) \\ &= x^m\left(\frac{2\pi}{n}\right)^{1/2}\left(1+\mathcal{O}(n^{-\epsilon})\right) \end{aligned} \tag{B.52}$$

for some $0<\epsilon<\frac{1}{2}$ and n large enough. We thus find for x as in (B.50)

$$h_{nm}^{(\gamma,\beta)} \sim \sqrt{\frac{n!}{m!}}\frac{2^{\frac{n-m}{2}}\left(\frac{n-m}{2}\right)!(x^2-1)^{\frac{n-m}{2}}}{(n-m)!\pi}x^m\sqrt{\frac{2\pi}{n}}\,. \tag{B.53}$$

Stirling's formula $n! \sim n^n \,\mathrm{e}^{-n}$ can be used yielding

$$
\begin{aligned}
&\sqrt{n!}\frac{2^{\frac{n-m}{2}}\left(\frac{n-m}{2}\right)!}{(n-m)!} \\
\sim \quad & \exp\left(\frac{n-m}{2}\ln(n-m)-(n-m)\ln(n-m)-\frac{n-m}{2}+(n-m)+\frac{n}{2}\ln(n)-\frac{n}{2}\right) \\
\sim \quad & 1
\end{aligned}
\tag{B.54}
$$

which gives us the final result

$$
h_{nm}^{(\gamma,\beta)} \sim \frac{(x^2-1)^{n/2}}{\sqrt{n}} \sim n^{-1/2}\left|\frac{\beta-\gamma}{\beta+\gamma}\right|^n . \tag{B.55}
$$

□

The transition matrix possesses an exponential decay unless the angle between β and γ is less than $\pi/2$. Note that this ensures the pointwise convergence of the sum $\sum_{n=0}^{\infty} f_n^{(\gamma)}(q)h_{nm}^{(\gamma,\beta)}$ in these cases since

$$
\begin{aligned}
f_n^{(\gamma)}(q) \quad & \sim \quad \frac{\sqrt{n!}}{2^{n/2}(n/2)!}H_n(\sqrt{\gamma}\,q) \\
& \sim \quad \mathrm{e}^{\sqrt{2n}|\mathfrak{Im}(\sqrt{\gamma}\,q)|}
\end{aligned}
\tag{B.56}
$$

due to Stirling's formula and the asymptotic behavior of the Hermite polynomial (see equation (5.50)).

Appendix C

Asymptotics of Generalized Hermite Coefficients

In the following we will determine the asymptotics of the generalized Hermite coefficients of various classes of objects, as presented in section 6.2. We first introduce Gel'fand-Shilov spaces, whose asymptotics will be estimated from above afterwards. Tempered distributions are considered afterwards.

C.1 Gel'fand-Shilov Spaces

In the following we will give a brief account on *Gel'fand-Shilov* spaces. The Gel'fand-Shilov space $\mathcal{S}_\alpha^\beta(\mathbb{R})$ for some $\alpha, \beta \in \mathbb{R}_+$ is defined to be the space of smooth functions $\varphi(x) \in C^\infty(\mathbb{R})$, which obey the inequalities [GS64]

$$|x^k \varphi^{(q)}(x)| \leq C A^k B^q k^{k\alpha} q^{q\beta} \tag{C.1}$$

for all $x \in \mathbb{R}$ with constants $A, B, C > 0$ depending on φ and $k, q = 0, 1, 2, \ldots$.[1] The conditions pose restrictions on the behavior of the functions and its derivatives for $|x| \to \infty$. The smaller the parameters α and β are, the faster do the functions and their derivatives decay for $|x| \to \infty$. These spaces are non-trivial only for

$$\begin{aligned} &\alpha > 0\,,\ \beta > 0 \quad , \quad \alpha + \beta \geq 1 \\ &\alpha = 0\,,\ \beta > 1 \\ &\alpha > 1\,,\ \beta = 0 \quad . \end{aligned} \tag{C.2}$$

They are invariant under multiplication with polynomials and differentiation, while Fourier transformation interchanges α and β. For $\beta < 1$ the functions possess analytical continuations into the whole complex plane. A characterization which is equivalent to (C.1) is

$$|\varphi(x + \mathrm{i}\, y)| \leq C\, \mathrm{e}^{-a|x|^{1/\alpha} + b|y|^{1/(1-\beta)}} \tag{C.3}$$

where $a = \frac{\alpha}{\mathrm{e}\, A^{1/\alpha}}$ and $b > \frac{1-\beta}{\mathrm{e}} (B\, \mathrm{e})^{\frac{1}{1-\beta}}$.

A topology is given through the subspaces $\mathcal{S}_{\alpha,A}^{\beta,B}(\mathbb{R}) \subset \mathcal{S}_\alpha^\beta(\mathbb{R})$ consisting of all those functions which obey

$$|x^k \varphi^{(q)}(x)| \leq C \bar{A}^k \bar{B}^q k^{k\alpha} q^{q\beta} \tag{C.4}$$

for all $\bar{A} > A$ and $\bar{B} > B$. One then defines the set of norms on $\mathcal{S}_{\alpha,A}^{\beta,B}(\mathbb{R})$

$$\|\varphi\|_{\delta,\rho} = \sup_{x \in \mathbb{R}, k, p \in \mathbb{N}_0} \frac{|\varphi(x)|}{(A+\delta)^k (B+\rho)^q k^{\alpha k} q^{\beta q}}\,, \quad \delta, \rho = 1, \frac{1}{2}, \frac{1}{3}, \ldots \tag{C.5}$$

which defines a topology on these spaces. For $A_1 < A_2$ and $B_1 < B_2$ we have $\mathcal{S}_{\alpha,A_1}^{\beta,B_1}(\mathbb{R}) \subset \mathcal{S}_{\alpha,A_2}^{\beta,B_2}(\mathbb{R})$ and if $\{\varphi_n(x)\}$ is a convergent series in $\mathcal{S}_{\alpha,A_1}^{\beta,B_1}(\mathbb{R})$ it is also convergent in $\mathcal{S}_{\alpha,A_2}^{\beta,B_2}(\mathbb{R})$. The space $\mathcal{S}_\alpha^\beta(\mathbb{R})$ can then be

[1] The generalization to higher dimensions is straightforward but not important for us.

defined as the countable-infinite conjunction of all $\mathcal{S}^{\beta,B}_{\alpha,A}(\mathbb{R})$ with $A, B = 1, 2, \ldots$, and the topology on $\mathcal{S}^{\beta}_{\alpha}(\mathbb{R})$ is the induced limit topology. For $\alpha > 0, \beta > 0$ and $\alpha + \beta = 1$ these spaces are nontrivial only if $AB > \gamma$ for some $\gamma > 0$, where the admissible values for A and B are bounded from below by the hyperbola $AB = \gamma$. One can show that if $\varphi(x) \in \mathcal{S}^{\beta,B}_{\alpha,A}(\mathbb{R})$ then $\varphi(\lambda x) \in \mathcal{S}^{\beta,\lambda B}_{\alpha,A/\lambda}(\mathbb{R})$. Thus if the former space is nontrivial then also the latter.

Of special interest is the case $\alpha = \beta$ with $\alpha \in [1/2, 1]$ of quasi-analytic functions, which are subspaces of the space of entire functions on $\mathbb{C}$ restricted to $\mathbb{R}$. They are closed under Fourier transformation and form an algebra under the star-product, and might thus be a suitable test function space for noncommutative quantum field theories [Sol07b, Sol07a]. In [Sol10] it has been shown that every element in the multiplier algebra of $\mathcal{S}^{\alpha}_{\alpha}(\mathbb{R})$ can be approximated by functions in $\mathcal{S}^{\alpha}_{\alpha}(\mathbb{R})$ in the operator topology.[2]

The space $\mathcal{S}^{\alpha}_{\alpha}(\mathbb{R})$ have been characterized in terms of their Hermite coefficients [LCP07]. A function φ with Hermite coefficients $\{\varphi_n\}$ is in $\mathcal{S}^{\alpha}_{\alpha}(\mathbb{R})$ iff

$$\|\{\varphi_n\}\|_\theta = \left(\sum_{n=0}^{\infty} |\varphi_n|^2 \exp\left\{ 2\frac{\alpha}{\mathrm{e}} n^{\frac{1}{2\alpha}} \theta^{\frac{1}{\alpha}} \right\} \right)^{1/2} < \infty \tag{C.6}$$

for some $\theta > 0$.[3] One defines the spaces $s_{\alpha,\bar{\theta}}$, which consists of those sequences $\{\varphi_n\}$ with finite norm with respect to (C.6) for all $\theta > \bar{\theta}$. The sequences of *ultrafast fall-off* s_α are then defined as the inductive limit of the family of spaces $\{s_{\alpha,\theta}, \theta \in \mathbb{R}_+\}$.

The dual space $\mathcal{S}^{\alpha}_{\alpha}(\mathbb{R})'$ has a similar characterization in terms of Hermite coefficients. A distribution T is in $\mathcal{S}^{\alpha}_{\alpha}(\mathbb{R})'$ iff its Hermite coefficients $T_n = \langle T|\phi_n\rangle$ obey the relation

$$|T_n| < \exp\left\{ 2\frac{\alpha}{\mathrm{e}} n^{\frac{1}{2\alpha}} \theta^{\frac{1}{\alpha}} \right\} \tag{C.9}$$

for all $\theta > 0$. In the following we will use these asymptotics to estimate the asymptotic behavior of the generalized Hermite coefficients, i.e. the coefficients in the generalized matrix basis.

C.2 Asymptotics for Generalized Hermite Coefficients of Gel'fand-Shilov Functions

For a general application of the generalized matrix basis it is important to know, how the asymptotics of the generalized Hermite functions for various classes of functions and distributions look like. As an example, we pick the Gel'fand-Shilov space of type $\mathcal{S}^{\alpha}_{\alpha}(\mathbb{R})$ with $\alpha = 1/2$, which is dense in Schwartz space. We will show that (at least) the functions of a subset of it have Hermite coefficients with an exponential decay.

We determine bounds on the asymptotic behavior of the generalized Hermite coefficients $\psi^{(\vartheta)}_m = \langle f^{(\vartheta)}_m|\psi\rangle$ for $\psi \in \mathcal{S}^{1/2}_{1/2}(\mathbb{R})$ and $\psi \in \mathcal{S}'(\mathbb{R})$. Their corresponding Hermite coefficients ψ_m are characterized by the existence of a parameter $\theta > 0$ such that [LCP07]

$$\sum_{m=0}^{\infty} |\psi_m|\,\mathrm{e}^{\,m\theta} < \infty\,. \tag{C.10}$$

[2] This is actually true for all $\mathcal{S}^{\beta}_{\alpha}(\mathbb{R}^D)$ with $\beta \geq \alpha$.

[3] In [LCP07] the norm for any Gel'fand-Shilov space of *Romieu type* $\mathcal{S}^{\{M_p\}}$, where $\mathcal{S}^{\alpha}_{\alpha}$ is a special case of, is defined to be

$$\|\{\varphi_n\}\|_\theta = \left(\sum_{n=0}^{\infty} |\varphi_n|^2 \exp\left\{ 2M(\theta\sqrt{n}) \right\} \right)^{1/2} \tag{C.7}$$

with M the function

$$M(\theta\sqrt{n}) = \sup_{p\in\mathbb{N}_0} \log\left(\frac{(\theta\sqrt{n})}{M_p} \right). \tag{C.8}$$

In the case $\mathcal{S}^{\alpha}_{\alpha}(\mathbb{R})$ one has $(M_p)_{p\in\mathbb{N}_0} = (p^{\alpha p})_{p\in\mathbb{N}_0}$ and one can show that $M(\theta\sqrt{n}) = \frac{\alpha}{\mathrm{e}} n^{\frac{1}{2\alpha}} \theta^{\frac{1}{\alpha}}$.

We use the representation (B.11) of the transition matrix

$$
\begin{aligned}
h_{nm}^{(\gamma,\beta)} &= \left(\frac{\sqrt{\gamma}}{2^n n! \sqrt{\pi}}\right)^{1/2} \left(\frac{\sqrt{\beta}}{2^m m! \sqrt{\pi}}\right)^{1/2} \sqrt{\frac{2\pi}{\gamma+\beta}} \sqrt{\left(\frac{\gamma-\beta}{\gamma+\beta}\right)}^{m+n} \\
&\times \sum_{k=\max\left(0,\frac{n-m}{2}\right)}^{\lfloor n/2 \rfloor} \frac{n!m!(-1)^{\frac{m-n}{2}}(-1)^k}{k!\,(n-2k)!\,\left(k+\frac{m-n}{2}\right)!} \sqrt{\frac{16\beta\gamma}{(\beta-\gamma)^2}}^{n-2k} \\
&\times \begin{cases} 1\,, & \frac{|m-n|}{2} \in \mathbb{N} \\ 0\,, & \frac{|m-n|}{2} \in \mathbb{N}+\frac{1}{2} \end{cases} \\
&= \sum_{k=\max\left(0,\frac{n-m}{2}\right)}^{\lfloor n/2 \rfloor} G(m,k,n)\,. \qquad \text{(C.11)}
\end{aligned}
$$

for which in our case $\beta \in \mathbb{R}_+$. We give a bound on the generalized Hermite coefficient by the following estimation

$$
\begin{aligned}
|\psi_n^{(\gamma_\vartheta)}| &= |\sum_m^{\infty}{}' h_{nm}^{(\gamma,\beta)} \psi_m| \qquad \text{(C.12)} \\
&\leq \sum_m^{\infty}{}' \sum_{k=\max\left(0,\frac{n-m}{2}\right)}^{\lfloor n/2 \rfloor} |G(m,k,n)\psi_m| \qquad \text{(C.13)}
\end{aligned}
$$

where $\sum_n^{\infty}{}'$ denotes the sum over even or odd $m \geq 0$ depending on whether n is even or odd. This estimation affects the accuracy of the resulting bounds, since the k-sum of $h_{mn}^{(\gamma,\beta)}$ consists of terms with alternating sign. Better bounds might be found by keeping the k-sum within the modulus of $h_{mn}^{(\gamma,\beta)}$. We have to swap the summations. The following table should make clear which combinations (k, n) correspond to non-vanishing terms:

k	m								
0	–	–	…	–	–	n	$n+2$	$n+4$	…
1	–	–	…	–	$n-2$	n	$n+2$	$n+4$	…
⋮			⋱						
$\lfloor n/2 \rfloor$	$n-2k$	$n-2k+2$	…	$n-4$	$n-2$	n	$n+2$	$n+4$	…

For given k we can thus characterize the non-vanishing terms by $m = 2p + n - 2k$ and all integers $p \in \mathbb{N}_0$. For the case at hand, the sum over the different factors depending on n become

$$
\sum_m^{\infty}{}' \sum_{k=\max\left(0,\frac{n-m}{2}\right)}^{\lfloor n/2 \rfloor} |G(m,k,n)\psi_m| = \sum_{k=0}^{\lfloor n/2 \rfloor} \sum_{p=0}^{\infty} |G(2p+n-2k,k,n)\psi_{2p+n-2k}|\,. \qquad \text{(C.14)}
$$

In the following we denote $z = \frac{\gamma-\beta}{\gamma+\beta}$ and $r = |z|$. For simplicity we set $|\beta| = |\gamma|$ which implies $z = \mathrm{i}\tan(\vartheta/2)$ and thus

$$
\frac{16\beta\gamma}{(\beta-\gamma)^2} = 4(1-z^{-2}) = 4\frac{1+r^2}{r^2}\,. \qquad \text{(C.15)}
$$

We then have

$$
\begin{aligned}
& |G(m,k,n)\psi_m| \\
= & \left| C\left(\frac{n!}{2^n}\right)^{1/2} \sqrt{z}^{m+n} \sqrt{4\frac{1+r^2}{r^2}}^{n-2k} \frac{2^{-m/2}\sqrt{m!}(-1)^{\frac{m-n}{2}}(-1)^k}{k!\,(n-2k)!\,\left(k+\frac{m-n}{2}\right)!}\psi_m \right| \\
\overset{m\to 2p+n-2k}{=} & C\left(\frac{n!}{2^n}\right)^{1/2} \sqrt{r}^{n+2p+n-2k} \sqrt{4\frac{1+r^2}{r^2}}^{n-2k} \frac{2^{-p+k-n/2}\sqrt{(2p+n-2k)!}}{k!\,(n-2k)!p!} |\psi_{2p+n-2k}| \\
\overset{k\to\bar{k}=n/2-k}{=} & C\left(\frac{n!}{2^n}\right)^{1/2} \sqrt{r}^{n+2p+2\bar{k}} \sqrt{4\frac{1+r^2}{r^2}}^{2\bar{k}} \frac{2^{-p-\bar{k}}\sqrt{(2p+2\bar{k})!}(-1)^p}{(2\bar{k})!\,\left(\frac{n-2\bar{k}}{2}\right)!p!} |\psi_{2p+2\bar{k}}|
\end{aligned}
\tag{C.16}
$$

for some constant C. Since

$$
\pi^{1/2}\Gamma(2x) = 2^{2x-1}\Gamma(x)\Gamma(x+1/2) \tag{C.17}
$$

we have

$$
\frac{\sqrt{(2p+n-2k)!}}{p!} < \pi^{-1/2}2^{p+n/2-k}\frac{(p+\bar{k})!}{p!} = \pi^{-1/2}2^{p+n/2-k}(p+1)_{\bar{k}}\,. \tag{C.18}
$$

with $(p+1)_{\bar{k}}$ the Pochhammer symbol. The Hermite coefficients of the Gel'fand-Shilov function may not show a decay already at $m>0$ but only at $m>N$ for some finite N. Since for any given m the transition matrix decays exponentially for $n\to\infty$ these first $N/2$ terms of (C.12) can be neglected and we can safely assume $|\psi_{2p+2\bar{k}}| \sim \mathrm{e}^{-\theta(2p+2\bar{k})}$ for some $\theta>0$. We thus find

$$
\begin{aligned}
|\psi_n^{(\gamma_\vartheta)}| &< C\left(\frac{n!}{2^n}\right)^{1/2} r^{n/2}\sum_{k=0}^{\lfloor n/2\rfloor}\frac{1}{(2\bar{k})!\left(\frac{n-2\bar{k}}{2}\right)!}\left(4\,\mathrm{e}^{-2\theta}\frac{1+r^2}{r}\right)^{\bar{k}} \\
&\times \sum_{p=0}^{\infty}(p+1)_{\bar{k}}(r\,\mathrm{e}^{-2\theta})^p\,.
\end{aligned}
\tag{C.19}
$$

Using

$$
\sum_{p=0}^{\infty}(p+1)_{\bar{k}}(r\,\mathrm{e}^{-2\theta})^p = (1-r\,\mathrm{e}^{-2\theta})^{-\bar{k}}\bar{k}!\,. \tag{C.20}
$$

and

$$
\sum_{\bar{k}}^{\lfloor n/2\rfloor}\frac{k!(4y)^k}{(2k)!(\frac{n-2\bar{k}}{2})} = \frac{1}{(n/2)!}{}_2F_1\left(-\frac{n}{2},1;\frac{1}{2},-y\right) \tag{C.21}
$$

we find

$$
\begin{aligned}
|\psi_n^{(\gamma_\vartheta)}| &\sim \frac{2^{-n/2}\sqrt{n!}}{(n/2)!}r^{n/2}{}_2F_1\left(-\frac{n}{2},1;\frac{1}{2},-y\right) \\
&\sim r^{n/2}{}_2F_1\left(-\frac{n}{2},1;\frac{1}{2},-y\right)
\end{aligned}
\tag{C.22}
$$

with

$$
y = \frac{1+r^2}{r(\,\mathrm{e}^{2\theta}-r)}\,. \tag{C.23}
$$

Using 15.8.6 and 15.8.1 of [OLBC10], we find

$$
\begin{aligned}
r^{n/2}{}_2F_1\left(-\frac{n}{2},1;\frac{1}{2},-y\right) &= \frac{(1)_{n/2}}{(1/2)_{n/2}}r^{n/2}(1+y)^{n/2}{}_2F_1\left(-\frac{n}{2},-\frac{1}{2};-\frac{n}{2},\frac{y}{1+y}\right) \\
&= \frac{(1)_{n/2}}{(1/2)_{n/2}}r^{n/2}(1+y)^{n/2}\left(\frac{y}{1+y}\right)^{1/2}\,,
\end{aligned}
\tag{C.24}
$$

where ${}_2F_1(0,b,c) = 1$ has been used. The factor $(1)_{n/2}/(1/2)_{n/2}$ goes as $\sqrt{n/2}$ for large n. The asymptotic behavior is thus determined by the factor

$$\begin{aligned} r^{n/2}(1+y)^{n/2} &= \left(r + \frac{1+r^2}{\mathrm{e}^{2\theta} - r}\right)^{n/2} \\ &= \left(\frac{1+\mathrm{e}^{2\theta} r}{\mathrm{e}^{2\theta} - r}\right)^{n/2} . \end{aligned} \tag{C.25}$$

Note that for $r = 0$ we get back the exponential decay of the original Hermite coefficients. Though, in order to have a decay for a given $r \in [0,1]$, we have to restrict on those Gel'fand-Shilov functions for which

$$2\theta > \ln\left(\frac{1+r}{1-r}\right) \tag{C.26}$$

In the notation of section C.1 these functions form the space $s_{1/2,\bar{\theta}}$ for some $\bar{\theta}$ proportional to the rhs of (C.26). The larger we choose r the more do we have to restrict to Gel'fand-Shilov space. We emphasize that these bounds are not exact but rely on the estimation (C.13). The space of "good" functions might be larger than the one we found.

One should note that this estimation can not directly be applied to the dual space $(\mathcal{S}^{1/2}_{1/2}(\mathbb{R}))'$, which obeys equation (C.10) for all $\theta < 0$. A case which can be handled analogously is the space of tempered distribution, which will investigated in the next section.

C.3 Asymptotics for Tempered Distributions

Now we consider a tempered distributions $T \in \mathcal{S}(\mathbb{R})'$. We know that its Hermite coefficients T_m are bounded by

$$|T_m| < C(m+1)^q \tag{C.27}$$

for some constant C and all $q \in \mathbb{N}$. For technical reasons we will substitute $(m+1)^q$ by $(m+1)_q$, which is the Pochhammer symbol defined by

$$(m+1)_q = \frac{\Gamma(m+q+1)}{\Gamma(m+1)} . \tag{C.28}$$

We can then start at equation (C.20) in the previous section by substituting $(2p+2k+1)_q r^p$ for $(r\,\mathrm{e}^{-2\theta})^p$, where again $m = 2p + 2\bar{k}$ with $p \in \mathbb{N}_0$. The sum over p then gives

$$\begin{aligned} \sum_{p=0}^{\infty} (p+1)_{\bar{k}} (2p + 2\bar{k} + 1)_q r^p &\sim 2^q \sum_{p=0}^{\infty} (p+1)_{\bar{k}} (p + \bar{k} + 1)_q r^p \\ &= 2^q (1-r)^{-\bar{k}-q} k! (k+1)_q , \end{aligned} \tag{C.29}$$

which leads to the $\bar{k}$-sum

$$\sum_{\bar{k}=0}^{\lfloor n/2 \rfloor} \frac{\bar{k}!(k+1)_q}{(2\bar{k})!(\frac{n-2\bar{k}}{2})!} (4y')^{\bar{k}} = \frac{q!}{(n/2)!} {}_2F_1\left(-\frac{n}{2}, 1+q; \frac{1}{2}; -y'\right) . \tag{C.30}$$

with $y' = (1+r^2)/(r-r^2)$. Using 15.8.6 of [OLBC10] we find

$$\begin{aligned} & {}_2F_1\left(-\frac{n}{2}, 1+q; \frac{1}{2}; -y'\right) \\ = \; & \frac{(1+q)_{n/2}}{(1/2)_{n/2}} (1+y')^{n/2} {}_2F_1\left(-\frac{n}{2}, -\frac{1}{2} - q; -q - \frac{n}{2}; \frac{1}{1 - y'/4}\right) \\ = \; & \frac{(1+q)_{n/2}}{(1/2)_{n/2}} (1+y')^{n/2} \left(\frac{y'}{1+y'}\right)^{1/2+q} {}_2F_1\left(-q, -\frac{1}{2} - q; -q - \frac{n}{2}; -1/y'\right) \end{aligned} \tag{C.31}$$

The Hypergeometric function approaches 1 for large n and the asymptotic behavior is given by

$$\frac{(1+q)_{n/2}}{(1/2)_{n/2}} r^{n/2} \left(1+y'\right)^{n/2} \sim (n/2+q)^{q+1/2} \left(\frac{1+r}{1-r}\right)^{n/2} \tag{C.32}$$

which diverges exponentially for all $r \in (0,1)$. For $r = 0$ we get back the usual polynomial divergence of tempered distributions.

One could now ask the question, for which $r = \tan(\vartheta/2)$ it is still possible to find Gel'fand-Shilov functions $\psi \in \mathcal{S}^{1/2}_{1/2}(\mathbb{R})$ such that the series

$$\sum_{n=0}^{\infty} \psi_n^{(\vartheta)} \varphi_n^{(\vartheta)} \tag{C.33}$$

converges for all tempered distributions $\varphi \in \mathcal{S}'(\mathbb{R})$. Convergence requires

$$\psi_n^{(\vartheta)} \varphi_n^{(\vartheta)} \sim \left(\frac{1+r}{1-r}\right) \left(\frac{1+\mathrm{e}^{2\theta} r}{\mathrm{e}^{2\theta}-r}\right) < 1 \tag{C.34}$$

which can be rearranged to

$$\mathrm{e}^{2\theta} \quad > \quad \frac{1+\left(\frac{1-r}{1+r}\right) r}{\left(\frac{1-r}{1+r}\right) - r} = \frac{2-(r-1)^2}{2-(r+1)^2}\,. \tag{C.35}$$

This is only possible for $r + 1 < \sqrt{2}$ and thus for $\vartheta < \pi/4$. Again these results are only approximately, since the bounds on the asymptotics are only upper bounds and might not reflect the true asymptotic behavior of the corresponding Hermite sequences.

Appendix D

Expansion Theorem for Generalized Oscillator States

Using the results of appendix B we can now prove

Theorem D.1. *The linear span of the generalized oscillator functions $(f_n^{(\gamma_\vartheta)})_{n\in\mathbb{N}_0}$ is dense in $L^2(\mathbb{R})$.*

Proof: We will use the usual oscillator basis $(\phi_n)_{n\in\mathbb{N}_0}$ as an intermediate basis. Each $\psi \in L^2(\mathbb{R})$ can be approximated pointwise by the limit $\lim_{N\to\infty}\sum_{n=0}^{N}\phi_n(q)\psi_n$, which means that the linear span of the usual oscillator functions is dense in $L^2(\mathbb{R})$. In turn, each oscillator function can be expanded in the $f_n^{(\gamma)}$-basis for $\mathfrak{Re}(\gamma) > 0$, which is a corollary of the next lemma and proves that their span is also dense in $L^2(\mathbb{R})$. $\square$

More generally we prove that

Lemma D.2. *Let $\alpha, \beta \in \mathbb{C} - \{0\}$ with $\mathfrak{Re}(\alpha+\beta) > 0$ and*

$$0 < \left|\frac{\alpha-\beta}{\alpha+\beta}\right| < 1\,. \tag{D.1}$$

Then any function $f_m^{(\beta)}$ can be expanded in terms of $f_n^{(\alpha)}$, i.e. the following sum

$$f_m^{(\beta)}(q) \quad = \quad \sum_{n=0}^{\infty} f_n^{(\alpha)}(q)\, h_{nm}^{(\alpha,\beta)} \tag{D.2}$$

converges pointwise for every $q \in \mathbb{R}$, with transition matrix given by

$$h_{nm}^{(\alpha,\beta)} = \int_q f_n^{(\alpha)}(q)\, f_m^{(\beta)}(q)\,. \tag{D.3}$$

Proof: Due to proposition B.2 we are able to do the sum $\sum_n f_n^{(\gamma)}(x)\, h_{nm}^{(\gamma,\beta)}$ explicitly. Keeping in mind that each term with "$n-m$ odd" is zero we find

$$\begin{aligned}
&\sum_{n=0}^{\infty} f_n^{(\gamma)}(x)\, h_{nm}^{(\gamma,\beta)} \\
&= \quad {\sum_{n}^{\infty}}' \left(\frac{\sqrt{\gamma}}{2^n n!\sqrt{\pi}}\right)^{1/2} \mathrm{e}^{-\frac{\gamma}{2}x^2} H_n\left(\sqrt{\gamma}x\right) \\
&\times \quad \left(\frac{\sqrt{\gamma}}{2^n n!\sqrt{\pi}}\right)^{1/2} \left(\frac{\sqrt{\beta}}{2^m m!\sqrt{\pi}}\right)^{1/2} \sqrt{\frac{2\pi}{\beta+\gamma}} \sqrt{\left(\frac{\gamma-\beta}{\beta+\gamma}\right)}^{m+n} \\
&\times \quad \sum_{k=\max\left(0,\frac{m-n}{2}\right)}^{\lfloor m/2\rfloor} \frac{n!m!(-1)^k}{k!\,(m-2k)!\,\left(k+\frac{n-m}{2}\right)!} \sqrt{\frac{16\beta\gamma}{(\beta-\gamma)^2}}^{m-2k} \\
&= \quad \left(\frac{\sqrt{\beta}}{2^m m!\sqrt{\pi}}\right)^{1/2} {\sum_{n}^{\infty}}' \sum_{k=\max\left(0,\frac{m-n}{2}\right)}^{\lfloor m/2\rfloor} \frac{m!(-1)^k}{k!\,(m-2k)!} \sqrt{\frac{16\beta\gamma}{(\gamma-\beta)^2}}^{m-2k} \sqrt{\left(\frac{\gamma-\beta}{\beta+\gamma}\right)}^{m} \\
&\times \quad \sqrt{\frac{2\gamma}{\beta+\gamma}} \frac{1}{\left(k+\frac{n-m}{2}\right)!} \sqrt{\frac{1}{4}\left(\frac{\gamma-\beta}{\beta+\gamma}\right)}^{n} \mathrm{e}^{-\frac{\gamma}{2}x^2} H_n\left(\sqrt{\gamma}x\right)
\end{aligned} \tag{D.4}$$

where $\sum_n^{\infty}{}'$ denotes the sum over even or odd $n \geq 0$ depending on whether m is even or odd. We have to swap the summations. The double sum

$$\sum_n^{\infty}{}' \sum_{k=\max\left(0,\frac{m-n}{2}\right)}^{\lfloor m/2 \rfloor} . \tag{D.5}$$

can be rearranged such that k runs from 0 to $\lfloor m/2 \rfloor$. The following table should make clear which combinations (k, n) correspond to non-vanishing terms:

k	n								
0	–	–	...	–	–	m	$m+2$	$m+4$	...
1	–	–	...	–	$m-2$	m	$m+2$	$m+4$	...
$\vdots$			$\ddots$						
$\lfloor m/2 \rfloor$	$m-2k$	$m-2k+2$	...	$m-4$	$m-2$	m	$m+2$	$m+4$	...

For given k we can thus characterize the non-vanishing terms by $n = 2\ell + m - 2k$ and all integers $\ell \in \mathbb{N}_0$. For the case at hand, the sum over the different factors depending on n become

$$\begin{aligned}
&\sum_n^{\infty}{}' \sum_{k=\max\left(0,\frac{m-n}{2}\right)}^{\lfloor m/2 \rfloor} \frac{1}{(k+\frac{n-m}{2})!} G_n(x) \\
&= \sum_{k=0}^{\lfloor m/2 \rfloor} \sum_{\ell=0}^{\infty} \frac{1}{\ell!} G_{2l+m-2k}(x) .
\end{aligned} \tag{D.6}$$

For (D.4) this means

$$\begin{aligned}
&\sum_{n=0}^{\infty} f_n^{(\gamma)}(x)\, h_{nm}^{(\gamma,\beta)} \\
&= \left(\frac{\sqrt{\beta}}{2^m m! \sqrt{\pi}}\right)^{1/2} \sum_{k=0}^{\lfloor m/2 \rfloor} \frac{m!(-1)^k}{k!\,(m-2k)!} \sqrt{\frac{16\beta\gamma}{(\gamma-\beta)^2}}^{\,m-2k} \sqrt{\left(\frac{\gamma-\beta}{\beta+\gamma}\right)}^{\,m} \\
&\times \sqrt{\frac{2\gamma}{\beta+\gamma}} \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \sqrt{\frac{1}{4}\left(\frac{\gamma-\beta}{\beta+\gamma}\right)}^{\,2\ell+m-2k} \mathrm{e}^{-\frac{\gamma}{2}x^2} H_{2\ell+m-2k}\left(\sqrt{\gamma}x\right)
\end{aligned} \tag{D.7}$$

Here we can use equation (49.4.4) [Han75]

$$\sum_{k=0}^{\infty} \frac{t^k}{k!} H_{2k+p}(z) \;=\; (1+4t)^{-(p+1)/2} \exp\left(\frac{4tz^2}{1+4t}\right) H_p\left(\frac{z}{\sqrt{1+4t}}\right) , \; |t| < 1/4 \tag{D.8}$$

with the identification

$$z = \sqrt{\gamma}x , \quad p = m - 2k , \quad t = \frac{1}{4}\left(\frac{\gamma-\beta}{\beta+\gamma}\right) \tag{D.9}$$

and thus

$$\frac{4tz^2}{1+4t} = \frac{\gamma}{2}x^2 - \frac{\beta}{2}x^2 , \quad 1+4t = \frac{2\gamma}{\beta+\gamma} , \quad \frac{z}{\sqrt{1+4t}} = \sqrt{\frac{1}{2}(\beta+\gamma)}x . \tag{D.10}$$

We get

$$\begin{aligned}
&\sqrt{\frac{2\gamma}{\beta+\gamma}} \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \sqrt{\frac{1}{4}\left(\frac{\gamma-\beta}{\beta+\gamma}\right)}^{\,2\ell} \mathrm{e}^{-\frac{\gamma}{2}x^2} H_{2\ell+m-2k}\left(\sqrt{\gamma}x\right) \\
&= \sqrt{\frac{2\gamma}{\beta+\gamma}}^{\,-m+2k} \mathrm{e}^{-\frac{\beta}{2}x^2} H_{m-2k}\left(\sqrt{\frac{1}{2}(\beta+\gamma)}\,x\right) .
\end{aligned} \tag{D.11}$$

and (D.7) thus becomes

$$
\begin{aligned}
&\sum_{n=0}^{\infty} f_n^{(\gamma)}(x)\, h_{nm}^{(\gamma,\beta)} \\
&= \left(\frac{\sqrt{\beta}}{2^m m! \sqrt{\pi}}\right)^{1/2} \sum_{k=0}^{\lfloor m/2 \rfloor} \frac{m!(-1)^k}{k!\,(m-2k)!} \sqrt{\frac{16\beta\gamma}{(\gamma-\beta)^2}}^{\,m-2k} \sqrt{\left(\frac{\gamma-\beta}{\beta+\gamma}\right)}^{\,m} \\
&\quad \times \sqrt{\frac{2\gamma}{\beta+\gamma}}^{\,-m+2k} \sqrt{\frac{1}{4}\left(\frac{\gamma-\beta}{\beta+\gamma}\right)}^{\,m-2k} \mathrm{e}^{-\frac{\beta}{2}x^2} H_{m-2k}\left(\sqrt{\frac{1}{2}(\beta+\gamma)}\, x\right) \\
&= \left(\frac{\sqrt{\beta}}{2^m m! \sqrt{\pi}}\right)^{1/2} \sqrt{\frac{\beta-\gamma}{\beta+\gamma}}^{\,m} \\
&\quad \times \sum_{k=0}^{\lfloor m/2 \rfloor} \frac{m!}{k!\,(m-2k)!} \sqrt{\frac{2\beta}{\beta-\gamma}}^{\,m-2k} \mathrm{e}^{-\frac{\beta}{2}x^2} H_{m-2k}\left(\sqrt{\frac{1}{2}(\beta+\gamma)}\, x\right)
\end{aligned} \tag{D.12}
$$

Now we have to make a case study. For "m even" we will use equation (49.4.12) from [Han75][1]

$$
\sum_{k=0}^{n} \frac{(2n)!}{(2k)!(n-k)!}(-t)^k H_{2k}(z) = (-1-t)^n H_{2n}\left(z\left(1+\frac{1}{t}\right)^{-1/2}\right), \tag{D.14}
$$

for "m odd" equation (49.4.14) from [Han75]

$$
\sum_{k=0}^{n} \frac{(2n+1)!}{(2k+1)!(n-k)!}(-t)^k H_{2k+1}(z) = (-t)^n \left(1+\frac{1}{t}\right)^{n+\frac{1}{2}} H_{2n+1}\left(z\left(1+\frac{1}{t}\right)^{-1/2}\right) \tag{D.15}
$$

m even: Substituting in (D.12)

$$
\begin{aligned}
\ell &:= m/2 - k \\
k! &\rightarrow (m/2-\ell)! \\
(m-2k)! &\rightarrow (2\ell)!
\end{aligned} \tag{D.16}
$$

we get

$$
\left(\frac{\sqrt{\beta}}{2^m m! \sqrt{\pi}}\right)^{1/2} \mathrm{e}^{-\frac{\beta}{2}x^2} \sqrt{\frac{\beta-\gamma}{\beta+\gamma}}^{\,m} \sum_{\ell=0}^{m/2} \frac{m!}{(2\ell)!\,(m/2-\ell)!} \left[\frac{2\beta}{\beta-\gamma}\right]^{\ell} H_{2\ell}\left(\sqrt{\frac{1}{2}(\beta+\gamma)}\, x\right) \tag{D.17}
$$

Comparing to (D.14) we identify

$$
m/2 \;=\; n\,, \quad t = -\frac{2\beta}{\beta-\gamma}\,. \tag{D.18}
$$

Hence

$$
\begin{aligned}
\left(1+\frac{1}{t}\right)^{-1/2} &= \left(1-\frac{1}{2\beta}(\beta-\gamma)\right)^{-1/2} = \sqrt{\frac{1}{2}(\beta+\gamma)}^{\,-1} \\
-1-t &= \frac{-\beta-\gamma+2\beta}{\beta+\gamma} = \frac{\beta+\gamma}{\beta-\gamma}
\end{aligned} \tag{D.19}
$$

[1] The formulas (49.4.12) and (49.4.14) given in [Han75] are expressed in terms of the Pochhammer symbol $(-n)_k$. I used the relation

$$
(-n)_k \;=\; (-1)^k \frac{n!}{(n-k)!} \tag{D.13}
$$

to bring them into the given form (D.14) and (D.15).

and we find for "even m"

$$\begin{aligned}
\sum_{n=0}^{\infty} f_n^{(\gamma)}(x)\, h_{nm}^{(\gamma,\beta)} &= \left(\frac{\sqrt{\beta}}{2^m m! \sqrt{\pi}}\right)^{1/2} \mathrm{e}^{-\frac{\beta}{2}x^2} \sqrt{\frac{\beta-\gamma}{\beta+\gamma}}^{\,m} \\
&\quad \times \sum_{\ell=0}^{m/2} \frac{m!}{(2\ell)!\,(m/2-\ell)!} \left[\frac{2\beta}{\beta-\gamma}\right]^{\ell} H_{2\ell}\left(\frac{1}{2}(\beta+\gamma)\,x\right) \\
&= \left(\frac{\sqrt{\beta}}{2^m m! \sqrt{\pi}}\right)^{1/2} \mathrm{e}^{-\frac{\beta}{2}x^2} H_m(\sqrt{\beta}x) \\
&= f_m^{(\beta)}(x)
\end{aligned} \tag{D.20}$$

m odd: Substituting in (D.12)

$$\begin{aligned}
\ell &:= \frac{m-1}{2} - k \\
k! &\to \left(\frac{m-1}{2} - \ell\right)! \\
(m-2k)! &\to (2\ell+1)!
\end{aligned} \tag{D.21}$$

we get

$$\left(\frac{\sqrt{\beta}}{2^m m! \sqrt{\pi}}\right)^{1/2} \mathrm{e}^{-\frac{\beta}{2}x^2} \sqrt{\frac{\beta-\gamma}{\beta+\gamma}}^{\,m} \sqrt{\frac{2\beta}{\beta-\gamma}} \sum_{\ell=0}^{\frac{m-1}{2}} \frac{m!}{(2\ell+1)!\,\left(\frac{m-1}{2}-\ell\right)!} \left[\frac{2\beta}{\beta-\gamma}\right]^{\ell} H_{2\ell+1}\left(\frac{1}{2}(\beta+\gamma)\,x\right) \tag{D.22}$$

Comparing to (D.15) we identify

$$\frac{m-1}{2} = n\,, \quad t = -\frac{2\beta}{\beta-\gamma}\,. \tag{D.23}$$

Hence

$$(-t)^n \left(1+\frac{1}{t}\right)^{n+\frac{1}{2}} = \sqrt{\frac{\beta-\gamma}{2\beta}} \sqrt{\frac{\beta+\gamma}{\beta-\gamma}}^{\,m} \tag{D.24}$$

and we find for "odd m"

$$\begin{aligned}
\sum_{n=0}^{\infty} f_n^{(\gamma)}(x)\, h_{nm}^{(\gamma,\beta)} &= \left(\frac{\sqrt{\beta}}{2^m m! \sqrt{\pi}}\right)^{1/2} \mathrm{e}^{-\frac{\beta}{2}x^2} \sqrt{\frac{\beta-\gamma}{\beta+\gamma}}^{\,m} \sqrt{\frac{2\beta}{\beta-\gamma}} \\
&\quad \times \sum_{\ell=0}^{\frac{m-1}{2}} \frac{m!}{(2\ell+1)!\,\left(\frac{m-1}{2}-\ell\right)!} \left[\frac{2\beta}{\beta-\gamma}\right]^{\ell} H_{2\ell+1}\left(\sqrt{\frac{1}{2}(\beta+\gamma)}\,x\right) \\
&= \left(\frac{\sqrt{\beta}}{2^m m! \sqrt{\pi}}\right)^{1/2} \mathrm{e}^{-\frac{\beta}{2}x^2} H_m(x) \\
&= f_m^{(\beta)}(x)\,.
\end{aligned} \tag{D.25}$$

which proves the theorem. $\square$

Appendix E

Expression for the Generalized Landau Functions

Theorem (6.1)**.** *The generalized Landau functions $f_{mn}^{(E_\vartheta)}(\boldsymbol{x})$ with $m, n \in \mathbb{N}_0$ are given by*

$$\begin{aligned} f_{mn}^{(E_\vartheta)}(t,x) &= (-1)^{\min(m,n)} \sqrt{\frac{E}{\pi}} \sqrt{\frac{\min(m!,n!)}{\max(m!,n!)}} E_\vartheta^{|m-n|/2} \\ &\times \quad \mathrm{e}^{-\frac{E_\vartheta}{2} x_+^{(\vartheta)} x_-^{(\vartheta)}} (x_{-sgn(m-n)}^{(\vartheta)})^{|m-n|} \, L_{\min(m,n)}^{|m-n|} \left(E_\vartheta \, x_+^{(\vartheta)} x_-^{(\vartheta)} \right) \end{aligned} \tag{E.1}$$

with $x_\pm^{(\vartheta)} = t \pm \mathrm{i} \, \mathrm{e}^{-\mathrm{i}\,\vartheta} x$ and $L_n^\alpha(z)$ the generalized Laguerre Polynomials.

Proof: The generalized Landau functions are build on the generalized oscillator functions $f_m^{(\gamma_\vartheta)}$ with $\gamma = E/2$ and

$$f_{mn}^{(E_\vartheta)}(\boldsymbol{x}) = \sqrt{\frac{E}{4\pi}} \, \mathsf{W} \left[|f_m^{(\gamma_\vartheta)}\rangle\langle f_n^{(\gamma_{-\vartheta})}| \right] (\boldsymbol{x}) \, . \tag{E.2}$$

Using the definition of the generalized oscillator functions (6.17) we get

$$\begin{aligned} f_{mn}^{(E_\vartheta)}(t,x) &= \sqrt{\frac{\gamma}{2\pi}} \int_{\mathbb{R}} \mathrm{d}k \; \mathrm{e}^{\,\mathrm{i}\,\gamma\,k\,x} \, f_m^{(\gamma_\vartheta)}(t+k/2) \, f_n^{(\gamma_\vartheta)}(t-k/2) \\ &= \sqrt{\frac{\gamma}{2\pi}} \left(\frac{\gamma_\vartheta}{\pi} \right)^{1/2} \int_{\mathbb{R}} \mathrm{d}k \; \mathrm{e}^{\,\mathrm{i}\,\gamma\,k\,x} \, \mathrm{e}^{-\frac{1}{2}\gamma_\vartheta \left[(t+k/2)^2 + (t-k/2)^2 \right]} \left(\frac{1}{2^{m+n} m! n!} \right)^{1/2} \\ &\times \quad H_m \left(\sqrt{\gamma_\vartheta} \, (t+k/2) \right) H_n \left(\sqrt{\gamma_\vartheta} \, (t-k/2) \right) \, . \end{aligned} \tag{E.3}$$

The generating function of the Hermite polynomials

$$\mathrm{e}^{\,-a^2(\xi^2 - 2\xi q)} = \sum_{m=0}^{\infty} \frac{1}{m!} (a\xi)^m H_m(aq) \, . \tag{E.4}$$

will be used to obtain the generating function for the generalized matrix basis:

$$\begin{aligned} K^{(\gamma_\vartheta)}(\xi,\eta;t,x) &:= \sqrt{\frac{2\pi}{\gamma}} \sum_{m,n=0}^{\infty} \sqrt{\frac{2^{m+n}}{m!n!}} \left(\sqrt{\gamma_\vartheta} \xi \right)^m \left(\sqrt{\gamma_\vartheta} \eta \right)^n \, f_{mn}^{(E_\vartheta)}(t,x) \\ &= \left(\frac{\gamma_\vartheta}{\pi} \right)^{1/2} \int_{\mathbb{R}} \mathrm{d}k \; \mathrm{e}^{\,\mathrm{i}\,\gamma\,k\,x} \, \mathrm{e}^{-\frac{1}{2}\gamma_\vartheta \left[(t+k/2)^2 + (t-k/2)^2 \right]} \\ &\times \quad \mathrm{e}^{-\gamma_\vartheta \left(\xi^2 - 2\xi(t+k/2) + \eta^2 - 2\eta(t-k/2) \right)} \end{aligned} \tag{E.5}$$

The exponential is a Gaussian. Rearranging it yields

$$
\begin{aligned}
&-\frac{1}{4}\gamma_\vartheta\left[k^2+4k(\eta-\xi-\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\,\vartheta}x)+4(t^2+\xi^2+\eta^2-2\xi t-2\eta t)\right]\\
&= \quad -\frac{1}{4}\gamma_\vartheta\left[k+2(\eta-\xi-\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\,\vartheta}x)\right]^2\\
&\quad +\gamma_\vartheta\left[(\eta-\xi-\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\,\vartheta}x)^2-(t^2+\xi^2+\eta^2-2\xi t-2\eta t)\right]\\
&= \quad -\frac{1}{4}\gamma_\vartheta\left[k+2(\eta-\xi-\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\,\vartheta}x)\right]^2\\
&\quad +\gamma_\vartheta\left[\left(\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\,\vartheta}x\right)^2-t^2+2\eta\,\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\,\vartheta}x-2\xi\,\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\,\vartheta}x+2\xi t+2\eta t-2\eta\xi\right]\\
&= \quad -\frac{1}{4}\gamma_\vartheta\left[k+2(\eta-\xi-\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\,\vartheta}x)\right]^2+\gamma_\vartheta\left[-x_+x_-+2\xi x_-+2\eta x_+-2\eta\xi\right]\,. \qquad \text{(E.6)}
\end{aligned}
$$

where we rediscover the *generalized light cone coordinates* $x_\pm^{(\vartheta)}=t\pm\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\,\vartheta}x$ of section 6.3. One can see how the complex combinations $t\pm\mathrm{i}\,x$ used in the Euclidean setting for $\vartheta=0$ become light cone coordinates $t\pm x$ for $\vartheta=\pm\pi/2$. In the following we will simply write $x_\pm$ for the $x_\pm^{(\vartheta)}$. The k integration cancels the constant prefactor up to a factor of 2 leading to

$$
\begin{aligned}
K^{(\gamma_\vartheta)}(\xi,\eta;t,x) &= 2\,\mathrm{e}^{\gamma_\vartheta(-x_+x_-+2\xi x_-+2\eta x_+-2\eta\xi)}\\
&= \quad 2\,\mathrm{e}^{-\gamma_\vartheta x_+x_-}\sum_{k,\ell,p}^{\infty}\frac{1}{k!\ell!p!}\left(2\gamma_\vartheta x_-\xi\right)^k\left(2\gamma_\vartheta x_+\eta\right)^\ell\left(-2\gamma_\vartheta\eta\xi\right)^p\,. \qquad \text{(E.7)}
\end{aligned}
$$

The generalized matrix functions can now be obtained by taking suitable derivatives with respect to the variables ξ and η:

$$
f_{mn}^{(E_\vartheta)}(t,x) \quad = \quad \sqrt{\frac{|\gamma_\vartheta|}{2\pi}}\frac{1}{\sqrt{m!n!}}\left(\frac{1}{2\gamma_\vartheta}\right)^{\frac{m+n}{2}}\frac{\partial^m}{\partial\xi^m}\frac{\partial^n}{\partial\eta^n}K^{(\gamma_\vartheta)}(\xi,\eta;t,x)\Big|_{\xi=\eta=0}\,. \qquad \text{(E.8)}
$$

Let $m\geq n$. Then the derivatives of K are of the following form

$$
\begin{aligned}
&\frac{\partial^m}{\partial\xi^m}\frac{\partial^n}{\partial\eta^n}\sum_{k,\ell,p}\frac{1}{k!\ell!p!}a^kb^\ell c^p\xi^{k+p}\eta^{\ell+p}\Big|_{\xi=\eta=0}\\
&= \quad \sum_{k,\ell,p}\frac{1}{(m-p)!(\ell-p)!p!}a^kb^\ell c^p\frac{(k+p)!}{(k+p-m)!}\frac{(\ell+p)!}{(\ell+p-n)!}\delta_{0,k+p-m}\delta_{0,\ell+p-n}\\
&= \quad \sum_{p}^{n}a^{m-p}b^{n-p}c^p\frac{m!n!}{(m-p)!(\ell-p)!p!} \qquad \text{(E.9)}
\end{aligned}
$$

with $a=2\gamma_\vartheta x_-$, $b=2\gamma_\vartheta x_+$ and $c=-2\gamma_\vartheta$. This leads to

$$
\begin{aligned}
f_{mn}^{(E_\vartheta)}(t,x) \quad &= \quad \sqrt{\frac{2\gamma}{\pi}}\sqrt{m!n!}\,(2\gamma_\vartheta)^{\frac{m-n}{2}}\;\mathrm{e}^{-\gamma_\vartheta x_+x_-}x_-^{m-n}\\
&\quad\times\quad\sum_{p=0}^{n}\left(2\gamma_\vartheta x_+x_-\right)^{n-p}\frac{(-1)^p}{(m-p)!(\ell-p)!p!}\,. \qquad \text{(E.10)}
\end{aligned}
$$

This last sum can be identified with the associated Laguerre function by substituting $p\to n-p$ and

$$
L_n^k(y)=\sum_{q=0}^{n}\frac{(n+k)!(-1)^qy^q}{(n-q)!(k+q)!q!}\,. \qquad \text{(E.11)}
$$

We finally get

$$
f_{mn}^{(E_\vartheta)}(t,x) \quad = \quad (-1)^n\sqrt{\frac{2\gamma}{\pi}}\sqrt{\frac{n!}{m!}}\,(2\gamma_\vartheta)^{\frac{m-n}{2}}\;\mathrm{e}^{-\gamma_\vartheta x_+x_-}x_-^{m-n}L_n^{m-n}\left(2\gamma_\vartheta x_+x_-\right) \qquad \text{(E.12)}
$$

A similar calculation for $n \geq m$ leads to this result with $+ \leftrightarrow -$ and $m \leftrightarrow n$. Substituting $\gamma \to E/2$ solves this lemma. $\square$

The generalized Landau functions have certain symmetries which will be useful in appendix H.

Corollary E.1. *The generalized Landau functions given by equation* (E.1) *fulfill the relations*

$$\begin{aligned} f_{mn}^{(E_\vartheta)}(E^{-1}t, E^{-1}x) &= E f_{mn}^{(1/E_{-\vartheta})}(t,x) && \text{(E.13)} \\ f_{mn}^{(E_\vartheta)}(-t,x) &= (-1)^{m-n} f_{nm}^{(E_\vartheta)}(t,x) && \text{(E.14)} \\ f_{mn}^{(E_\vartheta)}(t,-x) &= f_{nm}^{(E_\vartheta)}(t,x) && \text{(E.15)} \\ f_{mn}^{(E_\vartheta)}(x,t) &= (-\mathrm{i})^{m-n} f_{nm}^{(E_{-\vartheta})}(t,x)\,. && \text{(E.16)} \end{aligned}$$

Proof: Equation (E.13) follows directly from the explicit expression (E.1) by noting that E and $x_\pm^{(\vartheta)}$ only occur in the combination $\sqrt{E}x_\pm^{(\vartheta)}$ and $Ex_+^{(\vartheta)}x_-^{(\vartheta)}$. The inversion of time $t \to -t$ only affects the term $x_{-sign(m-n)}^{(\vartheta)}$ with

$$x_{-sign(m-n)}^{(\vartheta)} \to -x_{+sign(m-n)}^{(\vartheta)} = -x_{-sign(n-m)}^{(\vartheta)}\,, \tag{E.17}$$

such that

$$f_{mn}^{(E_\vartheta)}(-t,x) = (-1)^{m-n} f_{nm}^{(E_\vartheta)}(t,x)\,. \tag{E.18}$$

The transformation $x \to -x$ yields

$$x_{-sign(m-n)}^{(\vartheta)} \to x_{+sign(m-n)}^{(\vartheta)} = x_{-sign(n-m)}^{(\vartheta)}\,, \tag{E.19}$$

which shows equation (E.15). Under the exchange of t and x we find

$$x_\pm^{(\vartheta)} = t \pm \mathrm{i}\,\mathrm{e}^{-\mathrm{i}\vartheta} x \to \pm\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\vartheta}(t \mp \mathrm{i}\,\mathrm{e}^{\mathrm{i}\vartheta} x) = \pm\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\vartheta} x_\mp^{(-\vartheta)} \tag{E.20}$$

and thus

$$\begin{aligned} \sqrt{E_\vartheta} x_\pm^{(\vartheta)} &\to \sqrt{E_{-\vartheta}}(\pm\mathrm{i}\, x_\mp^{(-\vartheta)}) \\ E_\vartheta x_+^{(\vartheta)} x_-^{(\vartheta)} &\to E_{-\vartheta} x_+^{(-\vartheta)} x_-^{(-\vartheta)}\,. \end{aligned} \tag{E.21}$$

Putting these into the expression of the generalized Landau function we find

$$\begin{aligned} f_{mn}^{(E_\vartheta)}(E^{-1}x, E^{-1}t) &= (-1)^{\min(m,n)} \sqrt{\frac{E}{\pi}} \sqrt{\frac{\min(m!,n!)}{\max(m!,n!)}} \sqrt{E_{-\vartheta}}^{|m-n|} \mathrm{e}^{-\frac{E_{-\vartheta}}{2} x_+^{(-\vartheta)} x_-^{(-\vartheta)}} \\ &\quad \times \left(-sgn(m-n)\,\mathrm{i}\, x_{+sgn(m-n)}^{(-\vartheta)}\right)^{|m-n|} L_{\min(m,n)}^{|m-n|}\left(E_{-\vartheta}\, x_+^{(-\vartheta)} x_-^{(-\vartheta)}\right) \\ &= (-\mathrm{i})^{m-n} f_{nm}^{(E_{-\vartheta})}(t,x) \end{aligned} \tag{E.22}$$

which proves the corollary. $\square$

Appendix F

Relative Probability to Create a Pair

We will now reconstruct a classical result in QED using the ϑ-regularization and the generalized Landau basis, namely the effective action for a complex KG field and a Dirac spinor in a classical external electric field. In his seminal paper [Sch51] Schwinger calculated the effective action for a Dirac field and a Klein-Gordon field in a constant, uniform, external electromagnetic background in 4 spacetime dimensions. In a pure electric field the one-loop correction to the Klein-Gordon field (before charge renormalization) is given by

$$\mathcal{L}^{(1)}_{\mathsf{KG}} = \frac{1}{16\pi^2}\int_0^\infty \mathrm{d}s\, s^{-3}\,\mathrm{e}^{-\mu^2 s}\left[eEs\frac{1}{\sin(eEs)} - 1\right], \tag{F.1}$$

while the Dirac case reads

$$\mathcal{L}^{(1)}_{\mathsf{D}} = -\frac{1}{8\pi^2}\int_0^\infty \mathrm{d}s\, s^{-3}\,\mathrm{e}^{-\mu^2 s}\left[eEs\cot(eEs) - 1\right]. \tag{F.2}$$

By shifting the contour above the real axis one picks up the poles $s = s_n = n\pi/eE$ by the residue theorem. The probability per unit time and unit volume to create a pair in the scalar theory is given by

$$2\mathfrak{Im}\mathcal{L}^{(1)}_{\mathsf{KG}} = \frac{\alpha^2}{2\pi^2}E^2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}\exp\left(-\frac{n\pi\mu^2}{eE}\right) \tag{F.3}$$

and for the fermionic case

$$2\mathfrak{Im}\mathcal{L}^{(1)}_{\mathsf{D}} = \frac{\alpha^2}{\pi^2}E^2\sum_{n=1}^{\infty}n^{-2}\exp\left(-\frac{n\pi\mu^2}{eE}\right). \tag{F.4}$$

We will now show that the regularized matrix basis approach leads to the same result quite effortless.

The generating functional of connected graphs $W[J, J^*]$ is defined via the vacuum-to-vacuum amplitude in presence of a source J

$$\langle\Omega, out|\Omega, in\rangle[J, J^*] = \mathrm{e}^{\,\mathrm{i}\,W[J,J^*]}, \tag{F.5}$$

with $|\Omega, in\rangle$ and $|\Omega, out\rangle$ the in- and out- vacua of the theory in presence of the external sources J and J^*. We first investigate the bosoninc case. In [Sch51] the following expression has been derived

$$W_{\mathsf{KG}}[J, J^*] = \iint J^*\Delta_c J - \mathrm{i}\,\ln\det\left(\Delta_F^{-1}\Delta_c\right). \tag{F.6}$$

with $\Delta_F = \Delta_c|_{E=0}$ the usual Feynman propagator. Thus using the ϑ-regularization we can write

$$W_{\mathsf{KG}}[J, J^*] = \iint J^*\Delta_c J - \mathrm{i}\,\ln\det\left(\frac{\partial_\mu^2 - \mu^2}{\mathsf{P}_\mu^2 - \mu^2}\right)_\epsilon, \tag{F.7}$$

which has to be understood in the limit $\epsilon \to 0$ with

$$\left(\frac{\partial_\mu^2 - \mu^2}{\mathsf{P}_\mu^2 - \mu^2}\right)_\epsilon = \left(\frac{\partial_\mu^2 - \mathrm{e}^{-\mathrm{i}\,\epsilon}\mu^2}{\mathrm{e}^{\,\mathrm{i}\,\epsilon}\mathsf{P}^2(\pi/2-\epsilon) - \mathrm{e}^{-\mathrm{i}\,\epsilon}\mu^2}\right). \tag{F.8}$$

The effective action is now defined as the Legendre transformed of $W_{\mathsf{KG}}[J, J^*]$ with respect to the *classical fields* $\phi_{cl}(\boldsymbol{x})$ and $\phi^*_{cl}(\boldsymbol{x})$

$$\Gamma_{\mathsf{KG}}[\phi_c, \phi^*_c] \quad = \quad W_{\mathsf{KG}}[J, J^*] - \int J\phi^*_c - \int J^*\phi_c \,, \tag{F.9}$$

where $\phi_{cl}(\boldsymbol{x})$ and $\phi^*_{cl}(\boldsymbol{x})$ are given by

$$\begin{aligned} \phi_{cl}(\boldsymbol{x}) &= \frac{\delta\, W[J, J^*]}{\delta\, J^*(\boldsymbol{x})} = \int_{\boldsymbol{x}'} \Delta_c(\boldsymbol{x}, \boldsymbol{x}')J(\boldsymbol{x}') \\ \phi^*_{cl}(\boldsymbol{x}) &= \frac{\delta\, W[J, J^*]}{\delta\, J(\boldsymbol{x})} = \int_{\boldsymbol{x}'} J^*(\boldsymbol{x}')\Delta_c(\boldsymbol{x}', \boldsymbol{x})\,. \end{aligned} \tag{F.10}$$

These may be inverted to give

$$J(\boldsymbol{x}) = (\mathsf{P}^2_\mu - \mu^2)_\epsilon \phi_{cl}(\boldsymbol{x}) \quad \text{and} \quad J^*(\boldsymbol{x}) = -(\mathsf{P}^2_\mu - \mu^2)_\epsilon \phi^*_{cl}(\boldsymbol{x})\,, \tag{F.11}$$

and inserting into (F.9) yields

$$\begin{aligned} \Gamma_{\mathsf{KG}}[\phi_{cl}, \phi^*_{cl}] \quad &= \quad \iint (\mathsf{P}^2_\mu - \mu^2)_\epsilon \phi^*_{cl}(\boldsymbol{x})\Delta(\boldsymbol{x}, \boldsymbol{x}')(\mathsf{P}^2_\mu - \mu^2)_\epsilon \phi_{cl}(\boldsymbol{x}) - \mathrm{i}\,\ln\det\left(\frac{\partial^2_\mu - \mu^2}{\mathsf{P}^2_\mu - \mu^2}\right)_\epsilon \\ &\quad - \int [(\mathsf{P}^2_\mu - \mu^2)_\epsilon \phi_{cl}]\phi^*_{cl} + \int [(\mathsf{P}^2_\mu - \mu^2)_\epsilon \phi^*_{cl}]\phi_{cl} \\ &= \quad \mathcal{S}_0[\phi_{cl}, \phi^*_{cl}] + \mathrm{i}\,\ln\det\left(\frac{\mathsf{P}^2_\mu - \mu^2}{\partial^2_\mu - \mu^2}\right)_\epsilon \end{aligned} \tag{F.12}$$

This is the *full* effective action of the theory, which means that the quantum content is completely given by the one-loop correction

$$\mathrm{i}\,\ln\det\left(\frac{\mathsf{P}^2_\mu - \mu^2}{\partial^2_\mu - \mu^2}\right)_\epsilon = W_{\mathsf{KG}}[0, 0]\,. \tag{F.13}$$

The one-loop correction in the Dirac case is given by inverting the functional determinant

$$\mathrm{i}\,\ln\det\left(\frac{\partial^2_\mu - \mu^2}{\mathsf{P}^2_\mu - \mu^2}\right)_\epsilon = W_{\mathsf{D}}[0, 0]\,. \tag{F.14}$$

In the following we will define $W_{\mathsf{KG/D}}[0, 0] =: W_{\mathsf{kg/D}} =: \int \mathrm{d}^4\boldsymbol{x}\, \mathcal{L}^{(1)}_{\mathsf{KG/D}}$, with $\mathcal{L}^{(1)}_{\mathsf{KG/D}}$ the one-loop effective Lagrangian. The probability that no pair gets produced out of the vacuum is given by

$$|\langle 0, out|0, in\rangle_{J=0}|^2 = \mathrm{e}^{-2\mathfrak{Im}W_{\mathsf{KG/D}}} \sim 1 - 2\mathfrak{Im}W_{\mathsf{KG/D}}\,, \tag{F.15}$$

and the relative probability to create a pair per unit time and unit volume is thus approximately given by $2\mathfrak{Im}\mathcal{L}_{\mathsf{KG/D}}$.[1] Since perturbation theory will always give real contributions, we see that pair production is a non-perturbative effect, given by the imaginary part of the generating functional.

Starting with the 4-dimensional regularized bosonic case, the effective action is given by

$$\begin{aligned} W_{\mathsf{KG}} \quad &= \quad \mathrm{i}\,\ln\det\left(\frac{\mathsf{P}^2_\mu - \mu^2}{\partial^2_\mu - \mu^2}\right)_\epsilon \\ &= \quad \mathrm{i}\,\mathrm{tr}\,\ln\left(\frac{\mathsf{P}^2_\mu - \mu^2}{\partial^2_\mu - \mu^2}\right)_\epsilon\,. \end{aligned} \tag{F.16}$$

[1] Of course in infinite time and in infinite volume there will be infinitely many pairs produced and $W_{\mathsf{KG/D}}$ will be infinite. This manifests itself in the $\boldsymbol{x}$-independence of $\mathcal{L}_{\mathsf{KG/D}}$, which is plausible, since the probability should not depend on time or position. Restricting to finite space V and a finite time interval T we have $W_{\mathsf{KG/D}} = TV\mathcal{L}_{\mathsf{KG/D}}$.

The operator $\mathrm{e}^{\,\mathrm{i}\,\epsilon}\mathsf{P}^2(\vartheta) - \mathrm{e}^{-\,\mathrm{i}\,\epsilon}\mu^2$ fulfills the eigenvalue equation

$$\left(\mathrm{e}^{\,\mathrm{i}\,\epsilon}\mathsf{P}^2(\vartheta) - \mathrm{e}^{-\,\mathrm{i}\,\epsilon}\mu^2\right) f^{(\vartheta)}_{mn}(\boldsymbol{x}) = \left(\mathrm{i}\,4E\left(m+\frac{1}{2}\right) - \mathrm{e}^{-\,\mathrm{i}\,\epsilon}\mu^2\right) f^{(\vartheta)}_{mn}(\boldsymbol{x})\,. \tag{F.17}$$

We will simply write μ^2 instead $\mathrm{e}^{-\,\mathrm{i}\,\epsilon}\mu^2$, keeping in mind that μ^2 is slightly imaginary. Additionally we will adhere to Schwinger's convention by substituting $E \to eE/2$. With the identity

$$\ln\left(\frac{a}{b}\right) = \int_0^\infty \frac{\mathrm{d}s}{s}\left(\mathrm{e}^{\,\mathrm{i}\,sa} - \mathrm{e}^{\,\mathrm{i}\,sb}\right) \tag{F.18}$$

which is valid for $\mathfrak{Im}(a) > 0$ and $\mathfrak{Im}(b) > 0$, the effective Lagrangian can be obtained by

$$\begin{aligned}
\mathcal{L}^{(1)}_{\mathsf{KG}}(\boldsymbol{x}) &= \mathrm{i}\,\langle\boldsymbol{x}|\ln\left(\frac{\mathsf{P}^2_\mu - \mu^2}{\partial^2_\mu - \mu^2}\right)_\epsilon|\boldsymbol{x}\rangle \\
&= \mathrm{i}\int\frac{\mathrm{d}s}{s}\int\frac{\mathrm{d}^2\boldsymbol{p}_\perp}{(2\pi)^2}\,\mathrm{e}^{-\,\mathrm{i}\,s\mu^2}\left(\sum_{mn} f^{(\vartheta)}_{nm}(\boldsymbol{x})f^{(\vartheta)}_{mn}(\boldsymbol{x})\,\mathrm{e}^{-s2eE(m+\frac{1}{2})} - \int\frac{\mathrm{d}^2\boldsymbol{p}_\parallel}{(2\pi)^2}\,\mathrm{e}^{\,\mathrm{i}\,s\boldsymbol{p}^2_\parallel}\right)\mathrm{e}^{\,\mathrm{i}\,s\boldsymbol{p}^2_\perp} \\
&= \mathrm{i}\int\frac{\mathrm{d}s}{s}\frac{\mathrm{d}^2\boldsymbol{p}_\perp}{(2\pi)^2}\,\mathrm{e}^{-\,\mathrm{i}\,s\mu^2}\left(\sum_{mn} f^{(\vartheta)}_{nm}(\boldsymbol{x})f^{(\vartheta)}_{mn}(\boldsymbol{x})\,\mathrm{e}^{-s2eE(m+\frac{1}{2})} - \frac{1}{4\pi s}\right)\mathrm{e}^{\,\mathrm{i}\,s\boldsymbol{p}^2_\perp}
\end{aligned} \tag{F.19}$$

where we denoted the momentum $p^\mu = (\boldsymbol{p}_\parallel, \boldsymbol{p}_\perp)$ with $\boldsymbol{p}_\perp$ denoting the momentum perpendicular to the electric field, while all scalar products involving $\boldsymbol{p}_\perp$ are understood to be Euclidean and those involving $\boldsymbol{p}_\parallel$ Minkowskian. We can now use corollary 7.2 to obtain

$$\begin{aligned}
&\mathrm{i}\int\frac{\mathrm{d}s}{s}\frac{\mathrm{d}^2\boldsymbol{p}_\perp}{(2\pi)^2}\,\mathrm{e}^{-\,\mathrm{i}\,s\mu^2}\left(\frac{eE}{2\pi}\,\mathrm{e}^{-seE}\sum_{m=0}^\infty \mathrm{e}^{-s2eEm} - \frac{1}{4\pi s}\right)\mathrm{e}^{\,\mathrm{i}\,s\boldsymbol{p}^2_\perp} \\
&= \frac{1}{16\pi^2}\int\frac{\mathrm{d}s}{s^2}\,\mathrm{e}^{-\,\mathrm{i}\,s\mu^2}\left(\frac{eE}{\sinh(eEs)} - \frac{1}{s}\right),
\end{aligned} \tag{F.20}$$

which is indeed independent of $\boldsymbol{x}$. The integral converges at infinity since μ^2 has a small imaginary part, and at 0 due to the $1/s$ subtraction of the free case. By deforming the integration contour as $s \mapsto -\,\mathrm{i}\,s$ this coincides with Schwinger's result (F.1).

The 4 dimensional spinor case can now be done in the same way, starting with

$$\begin{aligned}
W_{\mathsf{D}} &= -\,\mathrm{i}\,\ln\det\left(\frac{\not{\mathsf{P}} - \mu}{\mathrm{i}\,\not{\partial} - \mu}\right)_\epsilon \\
&= -\,\mathrm{i}\,\mathrm{tr}\ln\left(\frac{\not{\mathsf{P}} - \mu}{\mathrm{i}\,\not{\partial} - \mu}\right)_\epsilon.
\end{aligned} \tag{F.21}$$

The trace is understood to run over both spin and spacetime degrees of freedom. One uses

$$\begin{aligned}
\mathcal{L}^{(1)}_D &= -\,\mathrm{i}\,\mathrm{tr}\langle\boldsymbol{x}|\ln\left(\not{\mathsf{P}} - \mu\right)_\epsilon|\boldsymbol{x}\rangle \\
&= -\,\mathrm{i}\,\mathrm{tr}\langle\boldsymbol{x}|\frac{1}{2}\ln\left(\not{\mathsf{P}}^2 - \mu^2\right)_\epsilon|\boldsymbol{x}\rangle \\
&= -\,\mathrm{i}\,\mathrm{tr}\langle\boldsymbol{x}|\frac{1}{2}\ln\left(\mathsf{P}^2_\mu\mathbb{1} - \mu^2\mathbb{1} + \frac{\mathrm{i}}{2}\sigma_{\mu\nu}eF^{\mu\nu}\right)_\epsilon|\boldsymbol{x}\rangle
\end{aligned} \tag{F.22}$$

to boil down the spinor case to the scalar one. Here $F_{\mu\nu} = E\epsilon_{\mu\nu}$ is the electromagnetic field tensor and $\sigma_{\mu\nu} = \frac{\mathrm{i}}{2}[\gamma_\mu, \gamma_\nu]$. For the electric field case the matrix $\sigma_{\mu\nu}F^{\mu\nu}$ has the form

$$\frac{eE}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{F.23}$$

Each eigenvalue $\pm eE/2$ is two-fold degenerated, leading to an overall factor of two. Incorporated into our calculation this leads to

$$
\begin{aligned}
& -\mathrm{i}\int\frac{\mathrm{d}s}{s}\int\frac{\mathrm{d}^2\boldsymbol{p}_\perp}{(2\pi)^2}e^{-\mathrm{i}\,s\mu^2}\left(\frac{eE}{2\pi}\sum_{m=0}^{\infty}\left(\mathrm{e}^{-s2eE(m+\frac{1}{2}+\frac{1}{2})}+\mathrm{e}^{-s2eE(m+\frac{1}{2}-\frac{1}{2})}\right)-\frac{1}{s}\right)\mathrm{e}^{\,\mathrm{i}\,s\boldsymbol{p}_\perp^2} \\
= \quad & -\mathrm{i}\,\frac{1}{4\pi}\int\frac{\mathrm{d}s}{s^2}\,\mathrm{e}^{-\mathrm{i}\,s\mu^2}\left(\frac{eE}{2\pi}\sum_{m=0}^{\infty}\mathrm{e}^{-s2eEm}\left(\mathrm{e}^{-s2eE}+1\right)-\frac{1}{s}\right) \\
= \quad & -\frac{1}{8\pi^2}\int\frac{\mathrm{d}s}{s^2}e^{-\mathrm{i}\,s\mu^2}\left(eE\coth(eEs)-\frac{1}{s}\right). \qquad \text{(F.24)}
\end{aligned}
$$

By deforming the integration contour $s\mapsto -\mathrm{i}\,s$ one obtains Schwinger's result (F.2). This result supports the conjecture of the physical relevance of this regularization.

The matrix basis provides an easy way of doing the otherwise cumbersome calculations.

Appendix G

Proof of lemma 7.1

Lemma (7.1)**.** *Let $\boldsymbol{x} \in \mathbb{R}^2$ and $a \in \mathbb{C} - \{0\}$. The following identity holds*

$$\begin{aligned}\sum_{n=0}^{\infty} f_{mn}^{(E_\vartheta)}(\boldsymbol{x})\, f_{nm}^{(E_\vartheta)}(\boldsymbol{x}')a^n = \frac{E}{\pi} \exp\left\{-\frac{E}{2}\|\boldsymbol{x}-\boldsymbol{x}'\|_\vartheta^2 + (a-1)E(\boldsymbol{x},\boldsymbol{x}')_\vartheta - a\,\mathrm{i}\,\boldsymbol{x}\cdot\boldsymbol{E}\cdot\boldsymbol{x}'\right\} \\ \times \quad L_m\left(E\|\boldsymbol{x}-\boldsymbol{x}'\|_\vartheta^2 - a(1-a^{-1})^2E(\boldsymbol{x},\boldsymbol{x}')_\vartheta + (a-a^{-1})\,\mathrm{i}\,\boldsymbol{x}\cdot\boldsymbol{E}\cdot\boldsymbol{x}'\right)\,,\end{aligned} \tag{G.1}$$

where

$$(\boldsymbol{x},\boldsymbol{x}')_\vartheta = \cos(\vartheta)\,(\boldsymbol{x},\boldsymbol{x}')_E + \mathrm{i}\,\sin(\vartheta)\,(\boldsymbol{x},\boldsymbol{x}')_M \tag{G.2}$$

with $(\cdot,\cdot)_M$ the Minkowskian and $(\cdot,\cdot)_E$ the Euclidean scalar product and $\|\boldsymbol{x}-\boldsymbol{x}'\|_\vartheta^2 = (\boldsymbol{x}-\boldsymbol{x}',\boldsymbol{x}-\boldsymbol{x}')_\vartheta$.

Proof: In the case $m \geq n$ explicit expression for the first eigenfunction is

$$f_{mn}^{(E_\vartheta)}(\boldsymbol{x}) \quad = \quad (-1)^n\sqrt{\frac{E}{\pi}}\sqrt{\frac{n!}{m!}}\,\mathrm{e}^{-E_\vartheta x_+ x_-/2}(\sqrt{E_\vartheta}\,x_-)^{m-n}\,L_n^{m-n}\left(E_\vartheta\,x_+x_-\right)\,. \tag{G.3}$$

and a similar representation for the second factor

$$f_{nm}^{(E_\vartheta)}(\boldsymbol{x}') \quad = \quad (-1)^n\sqrt{\frac{E}{\pi}}\sqrt{\frac{n!}{m!}}\,\mathrm{e}^{-E_\vartheta x'_+ x'_-/2}(\sqrt{E_\vartheta}x'_+)^{m-n}\,L_n^{m-n}\left(E_\vartheta\,x'_+x'_-\right)\,. \tag{G.4}$$

with $x_\pm = t \pm \mathrm{i}\,\mathrm{e}^{-\mathrm{i}\,\vartheta}x$ and $E_\vartheta = \mathrm{e}^{\,\mathrm{i}\,\vartheta}E$. These representations can also be used for $n > m$ due to the identity

$$(-1)^n r^{m-n}L_n^{m-n}(r^2) = (-1)^m r^{n-m}\frac{m!}{n!}L_m^{n-m}(r^2)\,. \tag{G.5}$$

The sum over n thus has the form

$$\begin{aligned}\sum_{n=0}^{\infty} f_{mn}^{(E_\vartheta)}(\boldsymbol{x})\, f_{nm}^{(E_\vartheta)}(\boldsymbol{x}')a^n = \frac{E}{\pi}\frac{\left(E_\vartheta x_- x'_+\right)^m}{m!}\,\mathrm{e}^{-E_\vartheta\,x_+x_-/2 - E_\vartheta\,x'_+x'_-/2} \\ \times \quad \sum_{n=0}^{\infty} n!\left(\frac{a}{E_\vartheta x_- x'_+}\right)^n L_n^{m-n}(E_\vartheta\,x_+x_-)L_n^{m-n}(E_\vartheta\,x'_+x'_-)\,.\end{aligned} \tag{G.6}$$

and can be done using the identity (48.23.11) from [Han75]

$$\sum_{n=0}^{\infty} n!c^n L_n^{m-n}(\xi)L_n^{k-n}(\eta) = k!\,\mathrm{e}^{\,c\,\xi\,\eta}(1-\eta\,c)^{m-k}c^m L_k^{m-k}\left(\frac{(1-\xi\,c)(\eta\,c-1)}{c}\right) \tag{G.7}$$

for $k = m$, $\xi = E_\vartheta\,x_+x_-$, $\eta = E_\vartheta\,x'_+x'_-$ and $c = a/(E_\vartheta\,x_-x'_+)$. We get

$$\sum_{n=0}^{\infty} f_{mn}^{(E_\vartheta)}(\boldsymbol{x})\, f_{nm}^{(E_\vartheta)}(\boldsymbol{x}')a^n = \frac{E}{\pi}\,\mathrm{e}^{-\xi/2-\eta/2}\,\mathrm{e}^{c\xi\eta}t^m L_m\left(\eta+\xi-c\xi\eta-c^{-1}\right)\,. \tag{G.8}$$

The different combinations of $x_\pm$ and $x'_\pm$ can be written as

$$\begin{aligned}
\xi/2+\eta/2 &= E_\vartheta x_+ x_-/2 + E_\vartheta x'_+ x'_-/2 \\
&= \frac{E_\vartheta}{2}(x_+ - x'_+)(x_- - x'_-) + \frac{E_\vartheta}{2}(x_+ x'_- + x_- x'_+) \\
&= \frac{E}{2}\|\boldsymbol{x}-\boldsymbol{x}'\|^2_\vartheta + E(\boldsymbol{x},\boldsymbol{x}')_\vartheta
\end{aligned} \tag{G.9}$$

where we defined

$$\begin{aligned}
\mathrm{e}^{\,\mathrm{i}\,\vartheta}(x'_+ - x_+)(x'_- - x_-) &= \mathrm{e}^{\,\mathrm{i}\,\vartheta}(t'-t+\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\,\vartheta}[x'-x])(t'-t-\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\,\vartheta}[x'-x]) \\
&= \mathrm{e}^{\,\mathrm{i}\,\vartheta}(t-t')^2 + \mathrm{e}^{-\mathrm{i}\,\vartheta}(x-x')^2 \\
&= \cos(\vartheta)(\boldsymbol{x}-\boldsymbol{x}')^2_i + \mathrm{i}\,\sin(\vartheta)(\boldsymbol{x}-\boldsymbol{x}')^2_\mu \\
&=: \|\boldsymbol{x}-\boldsymbol{x}'\|^2_\vartheta
\end{aligned} \tag{G.10}$$

and

$$\begin{aligned}
\frac{1}{2}\mathrm{e}^{\,\mathrm{i}\,\vartheta}(x_+ x'_- + x_- x'_+) &= \frac{1}{2}\mathrm{e}^{\,\mathrm{i}\,\vartheta}(tt' + \mathrm{e}^{-2\,\mathrm{i}\,\vartheta}xx' - \mathrm{i}\,\mathrm{e}^{-\mathrm{i}\,\vartheta}(tx'-xt')) \\
&\quad + \frac{1}{2}\mathrm{e}^{\,\mathrm{i}\,\vartheta}(tt' + \mathrm{e}^{-2\,\mathrm{i}\,\vartheta}xx' + \mathrm{i}\,\mathrm{e}^{-\mathrm{i}\,\vartheta}(tx'-xt')) \\
&= \cos(\vartheta)\,(tt'+xx') + \mathrm{i}\,\sin(\vartheta)\,(tt'-xx') \\
&=: (\boldsymbol{x},\boldsymbol{x}')_\vartheta\,.
\end{aligned} \tag{G.11}$$

In addition we have

$$\begin{aligned}
c\eta\xi &= aE_\vartheta x_+ x'_- \\
&= aE_\vartheta(tt' + \mathrm{e}^{-2\,\mathrm{i}\,\vartheta}xx' - \mathrm{i}\,\mathrm{e}^{-\mathrm{i}\,\vartheta}(tx'-xt')) \\
&= a\left(E(\boldsymbol{x},\boldsymbol{x}')_\vartheta - \mathrm{i}\,\boldsymbol{x}\cdot\boldsymbol{E}\cdot\boldsymbol{x}'\right)\,.
\end{aligned} \tag{G.12}$$

and

$$\begin{aligned}
c^{-1} &= a^{-1}E_\vartheta x_- x'_+ \\
&= a^{-1}E_\vartheta(tt' + \mathrm{e}^{-2\,\mathrm{i}\,\vartheta}xx' + \mathrm{i}\,\mathrm{e}^{-\mathrm{i}\,\vartheta}(tx'-xt')) \\
&= a^{-1}\left(E(\boldsymbol{x},\boldsymbol{x}')_\vartheta + \mathrm{i}\,\boldsymbol{x}\cdot\boldsymbol{E}\cdot\boldsymbol{x}'\right)\,.
\end{aligned} \tag{G.13}$$

Piecing all parts together gives the desired expression

$$\begin{aligned}
&\sum_{n=0}^{\infty} f^{(E_\vartheta)}_{mn}(\boldsymbol{x})\, f^{(E_\vartheta)}_{nm}(\boldsymbol{x}')a^n \\
=\;& \frac{E}{\pi}\exp\left\{-\frac{E}{2}\|\boldsymbol{x}-\boldsymbol{x}'\|^2_\vartheta + (a-1)E(\boldsymbol{x},\boldsymbol{x}')_\vartheta + a\,\mathrm{i}\,\boldsymbol{x}'\cdot\boldsymbol{E}\cdot\boldsymbol{x}\right\} \\
\times\;& a^m\,L_m\left(E\|\boldsymbol{x}-\boldsymbol{x}'\|^2_\vartheta - a(1-a^{-1})^2E(\boldsymbol{x},\boldsymbol{x}')_\vartheta - (a-a^{-1})\,\mathrm{i}\,\boldsymbol{x}'\cdot\boldsymbol{E}\cdot\boldsymbol{x}\right)\,.
\end{aligned} \tag{G.14}$$

□

Appendix H

Fourier Transformed Matrix Functions

We need the Fourier transformation of the Landau functions as well as the generalized Landau functions. Though the ordinary Landau functions are a special case of their generalizations, we have to distinguish both cases, due to different signatures the Fourier transformation depends on in the different spaces. We begin with Euclidean Fourier transformation of the ordinary Landau functions:

Theorem H.1. *The Euclidean Fourier transformation of $f^{(B)}_{mn}(\boldsymbol{x})$ is given by*

$$\mathcal{F}[f^{(B)}_{mn}](\boldsymbol{k}) = f^{(1/B)}_{mn}(\boldsymbol{k}) = \frac{\mathrm{i}^{\,m-n}}{B} f^{(B)}_{mn}(\tilde{\boldsymbol{k}})\,. \tag{H.1}$$

with $\tilde{\boldsymbol{k}} = \boldsymbol{B}^{-1}\cdot\boldsymbol{k} = B^{-1}(-k^2, k^1)$.

Proof: The Euclidean wave operators are given by

$$\begin{aligned}\mathsf{P}_i^2 &= -(\partial_1^2+\partial_2^2) - 2\,\mathrm{i}\,B(x^2\partial^1 - x^1\partial^2) + B^2(x_1^2+x_2^2)\\ \tilde{\mathsf{P}}_i^2 &= -(\partial_1^2+\partial_2^2) + 2\,\mathrm{i}\,B(x^2\partial^1 - x^1\partial^2) + B^2(x_1^2+x_2^2)\,.\end{aligned} \tag{H.2}$$

Denoting $\hat{\partial}_\mu = \partial/\partial k^\mu$, the operator P_i^2 has the following form in Fourier space

$$\begin{aligned}&\int_x (\mathsf{P}_i^2\phi)(\boldsymbol{x})\,\mathrm{e}^{-\,\mathrm{i}\,\boldsymbol{k}\cdot\boldsymbol{x}}\\ &= \int_x \phi(\boldsymbol{x})\,\tilde{\mathsf{P}}_i^2\,\mathrm{e}^{-\,\mathrm{i}\,\boldsymbol{k}\cdot\boldsymbol{x}}\\ &= \left((k_1^2+k_2^2) + 2\,\mathrm{i}\,B(k^1\hat{\partial}^2 - k^2\hat{\partial}^1) - B^2(\hat{\partial}_1^2+\hat{\partial}_2^2)\right)\hat{\phi}(\boldsymbol{k})\\ &= B^2\left(-(\hat{\partial}_1^2+\hat{\partial}_2^2) - 2\,\mathrm{i}\,B^{-1}(k^2\hat{\partial}^1 - k^1\hat{\partial}^2) + B^{-2}(k_1^2+k_2^2)\right)\hat{\phi}(\boldsymbol{k})\\ &= B^2\mathcal{P}_i^2\end{aligned} \tag{H.3}$$

where $\mathcal{P}_i^2$ has the same form as P_i^2 with $\partial_\mu \to \hat{\partial}_\mu$, $x^\mu \to k^\mu$ and $B \to B^{-1}$. On the other hand by substituting $\phi = f^{(B)}_{mn}$ we find

$$\mathcal{F}[\mathsf{P}_i^2 f^{(B)}_{mn}](\boldsymbol{k}) = 4B\left(m+\frac{1}{2}\right)\mathcal{F}[f^{(B)}_{mn}](\boldsymbol{k})\,. \tag{H.4}$$

Thus renaming $\boldsymbol{k}\to\boldsymbol{x}$ we find

$$\mathcal{P}_i^2\mathcal{F}[f^{(B)}_{mn}](\boldsymbol{x}) = 4B^{-1}\left(m+\frac{1}{2}\right)\mathcal{F}[f^{(B)}_{mn}](\boldsymbol{x})\,. \tag{H.5}$$

Due to the Parseval equation the Fourier transformed functions have the same normalization as the original ones, from which we conclude

$$\mathcal{F}[f^{(B)}_{mn}](\boldsymbol{k}) = f^{(1/B)}_{mn}(\boldsymbol{k})\,. \tag{H.6}$$

The relation $f^{(1/B)}_{mn}(\boldsymbol{k}) = \frac{\mathrm{i}^{\,m-n}}{B} f^{(B)}_{mn}(\tilde{\boldsymbol{k}})$ follows from

$$\tilde{\boldsymbol{k}} = \boldsymbol{B}^{-1}\cdot\boldsymbol{k} = B^{-1}(-k^2, k^1) \tag{H.7}$$

and the symmetry relations (E.13)-(E.16) derived in appendix E. □

Now we come to the Minkowskian case:

Theorem H.2. *The Fourier transformation of $f_{mn}^{(E_\vartheta)}(\boldsymbol{x})$ is given by*

$$\mathcal{F}[f_{mn}^{(E_\vartheta)}](\boldsymbol{k}) = f_{nm}^{(1/E_\vartheta)}(\boldsymbol{k}) = \frac{(-\mathrm{i})^{m-n}}{E} f_{mn}^{(E_\vartheta)}(\tilde{\boldsymbol{k}})\,. \tag{H.8}$$

with $\tilde{\boldsymbol{k}} = \boldsymbol{E}^{-1} \cdot \boldsymbol{k} = -E^{-1}(k^1, k^0)$.

Proof: In Minkowski spacetime the wave operators can be written as

$$\begin{aligned}
\mathsf{P}_i^2 &= -(\partial_0^2 + \partial_1^2) - 2\,\mathrm{i}\,E(x^1\partial^0 + x^0\partial^1) + E^2(x_0^2 + x_1^2)\,,\\
\mathsf{P}_\mu^2 &= -(\partial_0^2 - \partial_1^2) - 2\,\mathrm{i}\,E(x^1\partial^0 - x^0\partial^1) - E^2(x_0^2 - x_1^2)\,,\\
\tilde{\mathsf{P}}_i^2 &= -(\partial_0^2 + \partial_1^2) + 2\,\mathrm{i}\,E(x^1\partial^0 + x^0\partial^1) + E^2(x_0^2 + x_1^2)\,,\\
\tilde{\mathsf{P}}_\mu^2 &= -(\partial_0^2 - \partial_1^2) + 2\,\mathrm{i}\,E(x^1\partial^0 - x^0\partial^1) - E^2(x_0^2 - x_1^2)\,.
\end{aligned} \tag{H.9}$$

The regularized wave operators then have the form

$$\begin{aligned}
\mathsf{P}^2(\vartheta) &= \mathrm{e}^{\,\mathrm{i}\,\vartheta}\left(\cos(\vartheta)\mathsf{P}_i^2 - \mathrm{i}\,\sin(\vartheta)\mathsf{P}_\mu^2\right)\\
&= \mathrm{e}^{\,\mathrm{i}\,\vartheta}\left(-(\mathrm{e}^{-\,\mathrm{i}\,\vartheta}\partial_0^2 + \mathrm{e}^{\,\mathrm{i}\,\vartheta}\partial_1^2) - 2\,\mathrm{i}\,E(\mathrm{e}^{-i\vartheta}x^1\partial^0 + \mathrm{e}^{i\vartheta}x^0\partial^1) + (\mathrm{e}^{\,\mathrm{i}\,\vartheta}x_0^2 + \mathrm{e}^{-\,\mathrm{i}\,\vartheta}x_1^2)\right)\,,
\end{aligned} \tag{H.10}$$

$$\begin{aligned}
\tilde{\mathsf{P}}^2(\vartheta) &= \mathrm{e}^{\,\mathrm{i}\,\vartheta}\left(\cos(\vartheta)\tilde{\mathsf{P}}_i^2 - \mathrm{i}\,\sin(\vartheta)\tilde{\mathsf{P}}_\mu^2\right)\\
&= \mathrm{e}^{\,\mathrm{i}\,\vartheta}\left(-(\mathrm{e}^{-\,\mathrm{i}\,\vartheta}\partial_0^2 + \mathrm{e}^{\,\mathrm{i}\,\vartheta}\partial_1^2) + 2\,\mathrm{i}\,E(\mathrm{e}^{-i\vartheta}x^1\partial^0 + \mathrm{e}^{i\vartheta}x^0\partial^1) + (\mathrm{e}^{\,\mathrm{i}\,\vartheta}x_0^2 + \mathrm{e}^{-\,\mathrm{i}\,\vartheta}x_1^2)\right)\,.
\end{aligned} \tag{H.11}$$

In Fourier space we find

$$\begin{aligned}
&\int_x (\mathsf{P}_\mu^2\phi)(\boldsymbol{x})\,\mathrm{e}^{-\,\mathrm{i}\,\boldsymbol{k}\cdot\boldsymbol{x}}\\
&= \int_x \phi(\boldsymbol{x})\,\tilde{\mathsf{P}}_\mu^2\,\mathrm{e}^{-\,\mathrm{i}\,\boldsymbol{k}\cdot\boldsymbol{x}}\\
&= \left((k_0^2 - k_1^2) + 2\,\mathrm{i}\,E(k^0\hat\partial^1 - k^1\hat\partial^0) + E^2(\hat\partial_0^2 - \hat\partial_1^2)\right)\hat\phi(\boldsymbol{k})\\
&= -E^2\left(-(\hat\partial_0^2 - \hat\partial_1^2) + 2\,\mathrm{i}\,E^{-1}(k^1\hat\partial^0 - k^0\hat\partial^1) - E^{-2}(k_0^2 - k_1^2)\right)\hat\phi(\boldsymbol{k})
\end{aligned} \tag{H.12}$$

and

$$\begin{aligned}
&\int_x (\mathsf{P}_i^2\phi)(\boldsymbol{x})\,\mathrm{e}^{-\,\mathrm{i}\,\boldsymbol{k}\cdot\boldsymbol{x}}\\
&= \int_x \phi(\boldsymbol{x})\,\tilde{\mathsf{P}}_i^2\,\mathrm{e}^{-\,\mathrm{i}\,\boldsymbol{k}\cdot\boldsymbol{x}}\\
&= \left((k_0^2 + k_1^2) + 2\,\mathrm{i}\,E(k^0\hat\partial^1 + k^1\hat\partial^0) - E^2(\hat\partial_0^2 + \hat\partial_1^2)\right)\hat\phi(\boldsymbol{k})\\
&= E^2\left(-(\hat\partial_0^2 + \hat\partial_1^2) + 2\,\mathrm{i}\,E^{-1}(k^1\hat\partial^0 + k^0\hat\partial^1) - E^{-2}(k_0^2 + k_1^2)\right)\hat\phi(\boldsymbol{k})\,.
\end{aligned} \tag{H.13}$$

We thus find

$$\begin{aligned}
&\mathcal{F}[\mathsf{P}^2(\vartheta)\phi](\boldsymbol{k})\\
&= \mathrm{e}^{\,\mathrm{i}\,\vartheta}E^2\left(-(\mathrm{e}^{\,\mathrm{i}\,\vartheta}\hat\partial_0^2 + \mathrm{e}^{-\,\mathrm{i}\,\vartheta}\hat\partial_1^2) + 2\,\mathrm{i}\,E^{-1}(\mathrm{e}^{i\vartheta}k^1\hat\partial^0 + \mathrm{e}^{-i\vartheta}k^0\hat\partial^1) + (\mathrm{e}^{-\,\mathrm{i}\,\vartheta}k_0^2 + \mathrm{e}^{\,\mathrm{i}\,\vartheta}k_1^2)\right)\hat\phi(\boldsymbol{k})\\
&= \mathrm{e}^{2\,\mathrm{i}\,\vartheta}E^2\tilde{\mathcal{P}}^2(-\vartheta)\hat\phi(\boldsymbol{k})\,,
\end{aligned} \tag{H.14}$$

where $\tilde{\mathcal{P}}^2(-\vartheta)$ has the same form as $\tilde{\mathsf{P}}^2(-\vartheta)$ with $\partial_\mu \to \hat\partial_\mu$, $x^\mu \to k^\mu$ and $E \to E^{-1}$. On the other hand by substituting $\phi = f_{mn}^{(E_\vartheta)}$ we find

$$\mathcal{F}[\mathsf{P}^2(\vartheta)f_{mn}^{(E_\vartheta)}](\boldsymbol{k}) = 4E_\vartheta\left(m + \frac{1}{2}\right)\mathcal{F}[f_{mn}^{(E_\vartheta)}](\boldsymbol{k})\,. \tag{H.15}$$

thus renaming $\boldsymbol{k} \to \boldsymbol{x}$ we find

$$\tilde{\mathcal{P}}^2(-\vartheta)\mathcal{F}[f_{mn}^{(E_\vartheta)}](\boldsymbol{x}) = 4(E_\vartheta)^{-1}\left(m + \frac{1}{2}\right)\mathcal{F}[f_{mn}^{(E_\vartheta)}](\boldsymbol{x})\,. \tag{H.16}$$

Due to the Parseval equation the Fourier transformed functions have the same normalization as the original ones, from which we conclude

$$\mathcal{F}[f_{mn}^{(E_\vartheta)}](\boldsymbol{k}) = f_{nm}^{(1/E_\vartheta)}(\boldsymbol{k})\,. \tag{H.17}$$

The relation $f_{nm}^{(1/E_\vartheta)}(\boldsymbol{k}) = \frac{(-\mathrm{i})^{m-n}}{E} f_{mn}^{(E_\vartheta)}(\boldsymbol{E}^{-1} \cdot \boldsymbol{k})$ with

$$\boldsymbol{E}^{-1} \cdot \boldsymbol{k} = -E^{-1}(k^1, k^0) \tag{H.18}$$

follows from the symmetry relations (E.13)-(E.16). □

Appendix I

Position Space Propagator

Theorem (8.1). *The propagator of the regularized, general LSZ model in* 2n dimensions *is given by*

$$\begin{aligned}\Delta^{(\epsilon,\sigma)}(\boldsymbol{x},\boldsymbol{x}') = & -\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\,\vartheta}\frac{E}{2\pi}\int_0^\infty \mathrm{d}s\;\mathrm{e}^{-s\mu^2}\frac{1}{\sinh(2sE_{-\vartheta})}\exp\left\{-\frac{\sinh(2s\tilde{E}_{-\vartheta})}{\sinh(2sE_{-\vartheta})}\,\mathrm{i}\,\boldsymbol{x}_1\cdot\boldsymbol{E}\cdot\boldsymbol{x}_1'\right\}\\ \times\quad & \exp\left\{-\frac{1}{2}\coth(2sE_{-\vartheta})E(\|\boldsymbol{x}_1\|_\vartheta^2+\|\boldsymbol{x}_1'\|_\vartheta^2)+\frac{\cosh(2s\tilde{E}_{-\vartheta})}{\sinh(2sE_{-\vartheta})}E(\boldsymbol{x}_1,\boldsymbol{x}_1')_\vartheta\right\}\\ \times\quad & \prod_{k=2}^n\frac{B_k}{2\pi}\frac{1}{\sinh(2sB_k)}\exp\left\{-\frac{\sinh(2s\tilde{B}_k)}{\sinh(2sB_k)}\,\mathrm{i}\,\boldsymbol{x}_k\cdot\boldsymbol{B}_k\cdot\boldsymbol{x}_k'\right\}\\ \times\quad & \exp\left\{-\frac{1}{2}\coth(2sB_k)B_k(\|\boldsymbol{x}_k\|_0^2+\|\boldsymbol{x}_k'\|_0^2)+\frac{\cosh(2s\tilde{B}_k)}{\sinh(2sB_k)}B_k(\boldsymbol{x}_k,\boldsymbol{x}_k')_0\right\}.\end{aligned} \tag{I.1}$$

with $\vartheta=\pi/2-\epsilon>0$, $\tilde{E}=(2\sigma-1)E$, $\tilde{B}_k=(2\sigma-1)B_k$ *and*

$$(\boldsymbol{x},\boldsymbol{x}')_\vartheta=\cos(\vartheta)\,(\boldsymbol{x},\boldsymbol{x}')_E+\mathrm{i}\,\sin(\vartheta)\,(\boldsymbol{x},\boldsymbol{x}')_M \tag{I.2}$$

with $(\cdot,\cdot)_M$ *the Minkowskian and* $(\cdot,\cdot)_E$ *the Euclidean scalar product and* $\|\boldsymbol{x}-\boldsymbol{x}'\|_\vartheta^2=(\boldsymbol{x}-\boldsymbol{x}',\boldsymbol{x}-\boldsymbol{x}')_\vartheta$.

Proof: The coordinates are denoted by $\boldsymbol{x}=(x^0,\ldots,x^d)$ and $\boldsymbol{x}_k=(x^{2k-2},x^{2k-1})$ with $k=1,\ldots,n$. The propagator is given by

$$\begin{aligned}\Delta^{(\epsilon,\sigma)}(\boldsymbol{x},\boldsymbol{x}') &= \langle\boldsymbol{x}|\left[\sigma\,\mathrm{e}^{\,\mathrm{i}\,\epsilon}\mathsf{K}^2(\vartheta)+(1-\sigma)\,\mathrm{e}^{\,\mathrm{i}\,\epsilon}\tilde{\mathsf{K}}^2(\vartheta)-\mathrm{e}^{-\mathrm{i}\,\epsilon}\mu^2\right]^{-1}|\boldsymbol{x}'\rangle\\ &= \mathrm{e}^{-\mathrm{i}\,\epsilon}\langle\boldsymbol{x}|\left[\sigma\mathsf{P}^2(\vartheta)+(1-\sigma)\tilde{\mathsf{P}}^2(\vartheta)+\mathrm{e}^{2\,\mathrm{i}\,\vartheta}\sum_{k=1}^{n-1}(\sigma\mathsf{P}_{i,k}^2+(1-\sigma)\tilde{\mathsf{P}}_{i,k}^2)+\mathrm{e}^{2\,\mathrm{i}\,\vartheta}\mu^2\right]^{-1}|\boldsymbol{x}'\rangle\end{aligned} \tag{I.3}$$

with $\vartheta=\pi/2-\epsilon>0$, where the regularized wave operators fulfill the eigenvalue equations

$$\begin{aligned}(\sigma\mathsf{P}^2(\vartheta)+(1-\sigma)\tilde{\mathsf{P}}^2(\vartheta))f_{m_1n_1}^{(E_\vartheta)}(\boldsymbol{x}_1) &= 4E_\vartheta\left(\sigma m_1+(1-\sigma)n_1+\frac{1}{2}\right)f_{m_1n_1}^{(E_\vartheta)}(\boldsymbol{x}_1)\\ (\sigma(\mathsf{P}_i^2)_k+(1-\sigma)(\tilde{\mathsf{P}}_i^2)_k)f_{m_kn_k}^{(B_k)}(\boldsymbol{x}_k) &= 4B_k\left(\sigma m_k+(1-\sigma)n_k+\frac{1}{2}\right)f_{mn}^{(B_k)}(\boldsymbol{x}_k)\end{aligned} \tag{I.4}$$

with $f_{mn}^{(B_k)}(\boldsymbol{x}_k)$ the usual Landau functions and $B_k\in\mathbb{R}_+$. We set $\tilde{\sigma}=1-\sigma$. With the identity

$$a^{-1}=\int_0^\infty \mathrm{d}s\;\mathrm{e}^{-sa} \tag{I.5}$$

which is valid for $\mathfrak{Re}(a)>0$ we find

$$\begin{aligned}&\Delta^{(\epsilon,\sigma)}(\boldsymbol{x},\boldsymbol{x}')\\ =\;& -\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\,\vartheta}\int_0^\infty \mathrm{d}s\sum_{m_1n_1=0}^\infty f_{m_1n_1}^{(E_\vartheta)}(\boldsymbol{x}_1)f_{n_1m_1}^{(E_\vartheta)}(\boldsymbol{x}_1')\,\mathrm{e}^{-s\mu^2}\,\mathrm{e}^{-4sE\,\mathrm{e}^{-\mathrm{i}\,\vartheta}(\sigma m_1+\tilde{\sigma}n_1+1/2)}\\ \times\;& \prod_{k=2}^n\sum_{m_kn_k=0}^\infty f_{m_kn_k}^{(B_k)}(\boldsymbol{x}_k)f_{n_km_k}^{(B_k)}(\boldsymbol{x}_k')\,\mathrm{e}^{-4sB_k(\sigma m_k+\tilde{\sigma}n_k+1/2)}\end{aligned} \tag{I.6}$$

Using lemma (7.1) the sum over n_k gives the factor

$$\begin{aligned}
& \frac{B_k}{\pi} \sum_{m_k=0}^{\infty} \mathrm{e}^{-4sB_k\sigma(m_k+1/2)}\, \mathrm{e}^{-4sB_k m_k \tilde{\sigma}} \\
\times \quad & \exp\left\{-\frac{B_k}{2}\|\boldsymbol{x}_k-\boldsymbol{x}'_k\|_0^2+(\mathrm{e}^{-4sB_k\tilde{\sigma}}-1)B_k(\boldsymbol{x}_k,\boldsymbol{x}'_k)_0-\mathrm{e}^{-4sB_k\tilde{\sigma}}\ \mathrm{i}\,\boldsymbol{x}_k\cdot\boldsymbol{B}_k\cdot\boldsymbol{x}'_k\right\} \\
\times \quad & L_{m_k}\left(B_k\|\boldsymbol{x}_k-\boldsymbol{x}'_k\|_0^2-\mathrm{e}^{-4sB_k\tilde{\sigma}}(1-\mathrm{e}^{4sB_k\tilde{\sigma}})^2B_k(\boldsymbol{x}_k,\boldsymbol{x}'_k)_0+(\mathrm{e}^{-4sB_k\tilde{\sigma}}-\mathrm{e}^{4sB_k\tilde{\sigma}})\ \mathrm{i}\,\boldsymbol{x}_k\cdot\boldsymbol{B}_k\cdot\boldsymbol{x}'_k\right) \\
= \quad & \frac{B_k}{\pi} \sum_{m_k=0}^{\infty} \mathrm{e}^{-4sB_k(m_k+1/2)} \\
\times \quad & \exp\left\{-\frac{B_k}{2}\|\boldsymbol{x}_k-\boldsymbol{x}'_k\|_0^2+(\mathrm{e}^{-4sB_k\tilde{\sigma}}-1)B_k(\boldsymbol{x}_k,\boldsymbol{x}'_k)_0-\mathrm{e}^{-4sB_k\tilde{\sigma}}\ \mathrm{i}\,\boldsymbol{x}_k\cdot\boldsymbol{B}_k\cdot\boldsymbol{x}'_k\right\} \\
\times \quad & L_{m_k}\left(B_k\|\boldsymbol{x}_k-\boldsymbol{x}'_k\|_0^2-4\sinh(2sB_k\tilde{\sigma})^2B_k(\boldsymbol{x}_k,\boldsymbol{x}'_k)_0-2\sinh(4sB_k\tilde{\sigma})\ \mathrm{i}\,\boldsymbol{x}_k\cdot\boldsymbol{B}_k\cdot\boldsymbol{x}'_k\right)\ .
\end{aligned} \tag{I.7}$$

while the sum over n_1 gives

$$\begin{aligned}
& \frac{E}{\pi} \sum_{m_1=0}^{\infty} \mathrm{e}^{-4sE_{-\vartheta}(m_1+1/2)} \\
\times \quad & \exp\left\{-\frac{E}{2}\|\boldsymbol{x}_1-\boldsymbol{x}'_1\|_\vartheta^2+(\mathrm{e}^{-4sE_{-\vartheta}\tilde{\sigma}}-1)E(\boldsymbol{x}_1,\boldsymbol{x}'_1)_\vartheta-\mathrm{e}^{-4sE_{-\vartheta}\tilde{\sigma}}\ \mathrm{i}\,\boldsymbol{x}_1\cdot\boldsymbol{E}\cdot\boldsymbol{x}'_1\right\} \\
\times \quad & L_{m_1}\left(E\|\boldsymbol{x}_1-\boldsymbol{x}'_1\|_\vartheta^2-4\sinh(2sE_{-\vartheta}\tilde{\sigma})^2E(\boldsymbol{x}_1,\boldsymbol{x}'_1)_\vartheta-2\sinh(4sE_{-\vartheta}\tilde{\sigma})\ \mathrm{i}\,\boldsymbol{x}_1\cdot\boldsymbol{E}\cdot\boldsymbol{x}'_1\right)\ .
\end{aligned} \tag{I.8}$$

The sum over m_1 and m_k can be performed using equation (48.4.1) of [Han75]

$$\sum_{n=0}^{\infty} L_n(y)t^n = \frac{1}{1-t}\exp\left\{\frac{yt}{t-1}\right\} \quad , \quad |t|<1 \tag{I.9}$$

with $t=\mathrm{e}^{-4sB_k}$:

$$\begin{aligned}
& \frac{B_k}{\pi}\exp\left\{-\frac{B_k}{2}\|\boldsymbol{x}_k-\boldsymbol{x}'_k\|_0^2+(\mathrm{e}^{-4sB_k\tilde{\sigma}}-1)B_k(\boldsymbol{x}_k,\boldsymbol{x}'_k)_0-\mathrm{e}^{-4sB_k\tilde{\sigma}}\ \mathrm{i}\,\boldsymbol{x}_k\cdot\boldsymbol{B}_k\cdot\boldsymbol{x}'_k\right\} \\
\times \quad & \frac{\mathrm{e}^{-2sB_k}}{1-\mathrm{e}^{-4sB_k}}\exp\left\{\frac{\mathrm{e}^{-4sB_k}}{\mathrm{e}^{-4sB_k}-1}\left(B_k\|\boldsymbol{x}_k-\boldsymbol{x}'_k\|_0^2\right.\right. \\
& \left.\left.-4\sinh(2sB_k\tilde{\sigma})^2B_k(\boldsymbol{x}_k,\boldsymbol{x}'_k)_0-2\sinh(4sB_k\tilde{\sigma})\ \mathrm{i}\,\boldsymbol{x}_k\cdot\boldsymbol{B}_k\cdot\boldsymbol{x}'_k\right)\right\} \\
= \quad & \frac{B_k}{2\pi\sinh(2sB_k)}\exp\left\{-\frac{\cosh(2sB_k)}{2\sinh(2sB_k)}B_k\|\boldsymbol{x}_k-\boldsymbol{x}'_k\|_0^2\right. \\
+ \quad & \left[(\mathrm{e}^{-4sB_k\tilde{\sigma}}-1)+2\,\mathrm{e}^{-2sB_k}\frac{\sinh(2sB_k\tilde{\sigma})^2}{\sinh(2sB_k)}\right]B_k(\boldsymbol{x}_k,\boldsymbol{x}'_k)_0 \\
+ \quad & \left.\left[-\mathrm{e}^{-4sB_k\tilde{\sigma}}+\mathrm{e}^{-2sB_k}\frac{\sinh(4sB_k\tilde{\sigma})}{\sinh(2sB_k)}\right]\ \mathrm{i}\,\boldsymbol{x}_k\cdot\boldsymbol{B}_k\cdot\boldsymbol{x}'_k\right\}
\end{aligned} \tag{I.10}$$

and $t=\mathrm{e}^{-4sE_{-\vartheta}}$:

$$\begin{aligned}
& \frac{E}{2\pi\sinh(2sE_{-\vartheta})}\exp\left\{-\frac{\cosh(2sE_{-\vartheta})}{2\sinh(2sE_{-\vartheta})}E\|\boldsymbol{x}_1-\boldsymbol{x}'_1\|_\vartheta^2\right. \\
+ \quad & \left[(\mathrm{e}^{-4sE_{-\vartheta}\tilde{\sigma}}-1)+2\,\mathrm{e}^{-2sE_{-\vartheta}}\frac{\sinh(2sE_{-\vartheta}\tilde{\sigma})^2}{\sinh(2sE_{-\vartheta})}\right]E(\boldsymbol{x}_1,\boldsymbol{x}'_1)_\vartheta \\
+ \quad & \left.\left[-\mathrm{e}^{-4sE_{-\vartheta}\tilde{\sigma}}+\mathrm{e}^{-2sE_{-\vartheta}}\frac{\sinh(4sE_{-\vartheta}\tilde{\sigma})}{\sinh(2sE_{-\vartheta})}\right]\ \mathrm{i}\,\boldsymbol{x}_1\cdot\boldsymbol{E}\cdot\boldsymbol{x}'_1\right\}
\end{aligned} \tag{I.11}$$

The term proportional to $(\boldsymbol{x}_k, \boldsymbol{x}'_k)_0$ can be simplified using $\sinh^2(a) = \frac{1}{2}(\cosh(2a) - 1)$:

$$
\begin{aligned}
& (\,\mathrm{e}^{-4sB_k\tilde{\sigma}} - 1) + 2\,\mathrm{e}^{-2sB_k}\frac{\sinh(2sB_k\tilde{\sigma})^2}{\sinh(2sB_k)} \\
= \;& (\,\mathrm{e}^{-4sB_k\tilde{\sigma}} - 1) + \,\mathrm{e}^{-2sB_k}\frac{\cosh(4sB_k\tilde{\sigma})}{\sinh(2sB_k)} - \coth(2sB_k) + 1 \\
= \;& -\sinh(4sB_k\tilde{\sigma}) + \coth(2sB_k)\cosh(4sB_k\tilde{\sigma}) - \coth(2sB_k) \\
= \;& \frac{\cosh(2sB_k(1-2\tilde{\sigma})) - \cosh(2sB_k)}{\sinh(2sB_k)} \\
= \;& \frac{\cosh(2s\tilde{B}_k)}{\sinh(2sB_k)} - \frac{\cosh(2sB_k)}{\sinh(2sB_k)}
\end{aligned} \tag{I.12}
$$

where in the last step the addition theorem $\cosh(x-y) = \cosh(x)\cosh(y) - \sinh(x)\sinh(y)$ has been applied and $\tilde{B}_k := (1-2\tilde{\sigma})B_k = (2\sigma - 1)B_k$ has been defined. We find a similar result for terms proportional to $(\boldsymbol{x}_1, \boldsymbol{x}'_1)_\vartheta$:

$$
\begin{aligned}
& (\,\mathrm{e}^{-4sE_{-\vartheta}\tilde{\sigma}} - 1) + 2\,\mathrm{e}^{-2sE_{-\vartheta}}\frac{\sinh(2sE_{-\vartheta}\tilde{\sigma})^2}{\sinh(2sE_{-\vartheta})} \\
= \;& \frac{\cosh(2s\tilde{E}_{-\vartheta})}{\sinh(2sE_{-\vartheta})} - \frac{\cosh(2sE_{-\vartheta})}{\sinh(2sE_{-\vartheta})}
\end{aligned} \tag{I.13}
$$

with $\tilde{E}_{-\vartheta} := (1-2\tilde{\sigma})E_{-\vartheta} = (2\sigma-1)E_{-\vartheta}$. The triangle relation $\|\boldsymbol{x} - \boldsymbol{x}'\|^2_\vartheta = \|\boldsymbol{x}\|^2_\vartheta + \|\boldsymbol{x}'\|^2_\vartheta - 2(\boldsymbol{x}, \boldsymbol{x}')_\vartheta$ allows us to combine further terms

$$
\begin{aligned}
& -\frac{\cosh(2sB_k)}{2\sinh(2sB_k)}B_k\|\boldsymbol{x}_1 - \boldsymbol{x}'_1\|^2_0 + \left(\frac{\cosh(2s\tilde{B}_k)}{\sinh(2sB_k)} - \frac{\cosh(2sB_k)}{\sinh(2sB_k)}\right)B_k(\boldsymbol{x}_1, \boldsymbol{x}'_1)_0 \\
= \;& -\frac{\cosh(2sB_k)}{2\sinh(2sB_k)}B_k(\|\boldsymbol{x}_1\|^2_0 + \|\boldsymbol{x}'_1\|^2_0) + \frac{\cosh(2s\tilde{B}_k)}{\sinh(2sB_k)}B_k(\boldsymbol{x}_1, \boldsymbol{x}'_1)_0\,.
\end{aligned} \tag{I.14}
$$

and

$$
\begin{aligned}
& -\frac{\cosh(2sE_{-\vartheta})}{2\sinh(2sE_{-\vartheta})}E\|\boldsymbol{x}_1 - \boldsymbol{x}'_1\|^2_\vartheta + \left(\frac{\cosh(2s\tilde{E}_{-\vartheta})}{\sinh(2sE_{-\vartheta})} - \frac{\cosh(2sE_{-\vartheta})}{\sinh(2sE_{-\vartheta})}\right)E(\boldsymbol{x}_1, \boldsymbol{x}'_1)_\vartheta \\
= \;& -\frac{\cosh(2sE_{-\vartheta})}{2\sinh(2sE_{-\vartheta})}E(\|\boldsymbol{x}_1\|^2_\vartheta + \|\boldsymbol{x}'_1\|^2_\vartheta) + \frac{\cosh(2s\tilde{E}_{-\vartheta})}{\sinh(2sE_{-\vartheta})}E_{-\vartheta}(\boldsymbol{x}_1, \boldsymbol{x}'_1)_\vartheta\,.
\end{aligned} \tag{I.15}
$$

The terms proportional to $\,\mathrm{i}\,\boldsymbol{x}_k \cdot \boldsymbol{B}_k \cdot \boldsymbol{x}'_k$ can be rearranged to

$$
\begin{aligned}
& -\,\mathrm{e}^{-4sB_k\tilde{\sigma}} + \,\mathrm{e}^{-2sB_k}\frac{\sinh(4sB_k\tilde{\sigma})}{\sinh(2sB_k)} \\
= \;& \frac{-\cosh(4sB_k\tilde{\sigma})\sinh(2sB_k) + \sinh(4sB_k\tilde{\sigma})\sinh(2sB_k)}{\sinh(2sB_k)} \\
& + \;\frac{\cosh(2sB_k)\sinh(4sB_k\tilde{\sigma}) - \sinh(2sB_k)\sinh(4sB_k\tilde{\sigma})}{\sinh(2sB_k)} \\
= \;& -\frac{\sinh(2s\tilde{B}_k)}{\sinh(2sB_k)}
\end{aligned} \tag{I.16}
$$

where $\sinh(x-y) = \sinh(x)\cosh(y) - \cosh(x)\sinh(y)$ has been used, while $\,\mathrm{i}\,\boldsymbol{x}_1 \cdot \boldsymbol{E} \cdot \boldsymbol{x}'_1$ and can be vet in the same manner giving

$$
-\,\mathrm{e}^{-4sE_{-\vartheta}\tilde{\sigma}} + \,\mathrm{e}^{-2sE_{-\vartheta}}\frac{\sinh(4sE_{-\vartheta}\tilde{\sigma})}{\sinh(2sE_{-\vartheta})} = -\frac{\sinh(2s\tilde{E}_{-\vartheta})}{\sinh(2sE_{-\vartheta})}\,. \tag{I.17}
$$

Putting everything together we finally get

$$
\begin{aligned}
\Delta^{(\epsilon,\sigma)}(\boldsymbol{x},\boldsymbol{x}') = &-\mathrm{i}\,\mathrm{e}^{-2\,\mathrm{i}\,\vartheta}\,\frac{E}{2\pi}\int_0^\infty \mathrm{d}s\;\mathrm{e}^{-s\mu^2}\,\frac{1}{\sinh(2sE_{-\vartheta})}\exp\left\{-\frac{\sinh(2s\tilde{E}_{-\vartheta})}{\sinh(2sE_{-\vartheta})}\,\mathrm{i}\,\boldsymbol{x}_1\cdot\boldsymbol{E}\cdot\boldsymbol{x}_1'\right\}\\
&\times\quad \exp\left\{-\frac{1}{2}\coth(2sE_{-\vartheta})E(\|\boldsymbol{x}_1\|_\vartheta^2+\|\boldsymbol{x}_1'\|_\vartheta^2)+\frac{\cosh(2s\tilde{E}_{-\vartheta})}{\sinh(2sE_{-\vartheta})}E(\boldsymbol{x}_1,\boldsymbol{x}_1')_\vartheta\right\}\\
&\times\quad \prod_{k=2}^{n}\frac{B_k}{2\pi}\frac{1}{\sinh(2sB_k)}\exp\left\{-\frac{\sinh(2s\tilde{B}_k)}{\sinh(2sB_k)}\,\mathrm{i}\,\boldsymbol{x}_k\cdot\boldsymbol{B}_k\cdot\boldsymbol{x}_k'\right\}\\
&\times\quad \exp\left\{-\frac{1}{2}\coth(2sB_k)B_k(\|\boldsymbol{x}_k\|_0^2+\|\boldsymbol{x}_k'\|_0^2)+\frac{\cosh(2s\tilde{B}_k)}{\sinh(2sB_k)}B_k(\boldsymbol{x}_k,\boldsymbol{x}_k')_0\right\}\,.
\end{aligned}
\tag{I.18}
$$

□

Appendix J

Matrix Propagator

Theorem (8.5). *The matrix propagator for the $2n$ dimensional regularized LSZ model in Minkowski space-time is given by*

$$\begin{aligned}
&\Delta^{(\epsilon,\sigma)}_{\boldsymbol{m},\boldsymbol{m}+\boldsymbol{\alpha};\boldsymbol{\ell}+\boldsymbol{\alpha},\boldsymbol{\ell}} \\
&= -\,\mathrm{e}^{\,\mathrm{i}\,\epsilon}\frac{\theta}{8\Omega}\int_0^1 \mathrm{d}z\, z^{-\,\mathrm{i}\,\mathrm{e}^{\,\mathrm{i}\,\epsilon}(\sigma\alpha_1+1/2)+\sum_{i=2}^n(\sigma\alpha_i+1/2)-1+\frac{\theta\mu^2}{8\Omega}}\,\Delta^{(\epsilon)}_{n_1,n_1+\alpha_1;\ell_1+\alpha_1,\ell_1}\prod_{i=2}^n \Delta^{(E)}_{n_i,n_i+\alpha_i;\ell_i+\alpha_i,\ell_i}
\end{aligned} \tag{J.1}$$

with Minkowskian part

$$\begin{aligned}
&\Delta^{(\epsilon)}_{m,m+\alpha;\ell+\alpha,\ell} \\
&= \sum_{u=\max(0,-\alpha)}^{\min(m,\ell)} \frac{z^{-\,\mathrm{i}\,\mathrm{e}^{\,\mathrm{i}\,\epsilon}u}(1-z^{-i\,\mathrm{e}^{\,\mathrm{i}\,\epsilon}})^{m+\ell-2u}}{\left(1-\frac{(1-\Omega)^2}{(1+\Omega)^2}z^{-\,\mathrm{i}\,\mathrm{e}^{\,\mathrm{i}\,\epsilon}}\right)^{\alpha+m+\ell+1}}\left(\frac{4\Omega}{(1+\Omega)^2}\right)^{\alpha+2u+1}\left(\frac{1-\Omega}{1+\Omega}\right)^{m+\ell-2u}\mathcal{A}(m,\ell,\alpha,u)
\end{aligned} \tag{J.2}$$

and Euclidean part

$$\begin{aligned}
&\Delta^{(E)}_{m,m+\alpha;\ell+\alpha,\ell} \\
&= \sum_{u=\max(0,-\alpha)}^{\min(m,\ell)} \frac{z^{u}(1-z)^{m+\ell-2u}}{\left(1-\frac{(1-\Omega)^2}{(1+\Omega)^2}z\right)^{\alpha+m+\ell+1}}\left(\frac{4\Omega}{(1+\Omega)^2}\right)^{\alpha+2u+1}\left(\frac{1-\Omega}{1+\Omega}\right)^{m+\ell-2u}\mathcal{A}(m,\ell,\alpha,u)
\end{aligned} \tag{J.3}$$

where

$$\mathcal{A}(n,\ell,\alpha,u) = \sqrt{\begin{pmatrix}\alpha+n\\ \alpha+u\end{pmatrix}\begin{pmatrix}\alpha+\ell\\ \alpha+u\end{pmatrix}\begin{pmatrix}n\\ u\end{pmatrix}\begin{pmatrix}\ell\\ u\end{pmatrix}}. \tag{J.4}$$

and $\boldsymbol{\alpha} = (\alpha_1,\ldots,\alpha_n) \in \mathbb{Z}^n$ and $\alpha_i = n_i - m_i$.

Proof: The $2n$ dimensional, regularized LSZ wave operator in matrix basis is given by equation 6.94:

$$G^{(\epsilon,\sigma)}_{\boldsymbol{mn};\boldsymbol{k\ell}} = \mathrm{i}\,\mathcal{G}^{(\sigma)}_{m_1n_1;k_1\ell_1} - \mathrm{e}^{-\,\mathrm{i}\,\epsilon}\sum_{i=2}^n \mathcal{G}^{(\sigma)}_{m_in_i;k_i\ell_i} - \mathrm{e}^{-\,\mathrm{i}\,\epsilon}\mu^2 \tag{J.5}$$

with $\boldsymbol{m} = (m_1,\ldots,m_n), \boldsymbol{n} = (n_1,\ldots,n_n), \boldsymbol{k} = (k_1,\ldots,k_n), \boldsymbol{\ell} = (\ell_1,\ldots,\ell_n) \in \mathbb{N}^n$ and $\mathcal{G}^{(\sigma)}_{mn,k\ell}$ the two dimensional, massless, Euclidean LSZ matrix wave operators

$$\begin{aligned}
\mathcal{G}^{(\sigma)}_{mn;k\ell} &= \left(2\frac{\Omega^2+1}{\theta}(m+n+1)+\frac{4\tilde{\Omega}}{\theta}(n-m)\right)\delta_{m\ell}\,\delta_{n,k} \\
&\quad +2\frac{\Omega^2-1}{\theta}\left(\sqrt{nm}\delta_{m,\ell+1}\,\delta_{n,k+1}+\sqrt{(n+1)(m+1)}\delta_{m,\ell-1}\delta_{n,k-1}\right)
\end{aligned} \tag{J.6}$$

with frequencies $\Omega = E\theta_1/2 = B_i\theta_i/2$ and $\tilde{\Omega} = (2\sigma - 1)\Omega$. Each of these operators are nonzero only for

$$n_i - m_i = k_i - \ell_i =: \alpha_i \quad , \quad \forall\, i = 1, \ldots, n\,. \tag{J.7}$$

This is due to the $SO(1,1) \times SO(2)^{\times(n-1)}$-symmetry of the action. We can thus get rid of n parameters and write instead

$$G^{(\epsilon,\sigma)}_{\boldsymbol{m},\boldsymbol{m}+\boldsymbol{\alpha};\boldsymbol{\ell}+\alpha,\boldsymbol{\ell}} = \mathrm{i}\,\mathcal{G}^{(\sigma)}_{m_1,m_1+\alpha_1;\ell_1+\alpha_1,\ell_1} - \mathrm{e}^{-\mathrm{i}\,\epsilon}\sum_{i=2}^{n}\mathcal{G}^{(\sigma)}_{m_i,m_i+\alpha_i;\ell_i+\alpha_i,\ell_i} - \mathrm{e}^{-\mathrm{i}\,\epsilon}\mu^2\delta_{\boldsymbol{m}\boldsymbol{\ell}}\delta_{\boldsymbol{n}\boldsymbol{k}}\,. \tag{J.8}$$

with $\boldsymbol{\alpha} \in \mathbb{Z}^n$. The n parts of G are independent and its eigenfunctions are thus a product of the eigenfunctions of the individual $\mathcal{G}$'s. The mass term is already diagonal and also the terms proportional to $\tilde{\Omega}$. Thus for every α we are searching for solutions of the equations

$$\sum_{\ell=0}^{\infty}\mathcal{G}_{m,m+\alpha;\ell+\alpha,\ell}|_{\tilde{\Omega}=0}U^{(\alpha)}_{\ell v} \quad = \quad v\,U^{(\alpha)}_{mv}\,. \tag{J.9}$$

This equation has been solved in [GW05b] with the solutions given by

$$U^{(\alpha)}_{mv} = \sqrt{\begin{pmatrix}\alpha+m\\ m\end{pmatrix}\begin{pmatrix}\alpha+y\\ y\end{pmatrix}}\left(\frac{2\sqrt{\Omega}}{1+\Omega}\right)^{\alpha+1}\left(\frac{1-\Omega}{1+\Omega}\right)^{m+y}{}_2F_1\left(\begin{matrix}-m,-y\\ 1+\alpha\end{matrix}\,\middle|\, -\frac{4\Omega}{(1-\Omega)^2}\right)\,. \tag{J.10}$$

and eigenvalues

$$v = \frac{4\Omega}{\theta}(2y+\alpha+1)\,. \tag{J.11}$$

for $y = 0, 1, 2, \ldots$. The $\tilde{\Omega}$ term has to be added to the eigenvalues

$$v \to v' \quad = \quad \frac{4\Omega}{\theta}(2y + 2\sigma\alpha + 1)\,. \tag{J.12}$$

The complete matrix operator in $2n$ dimensions has the representation

$$G^{(\epsilon,\sigma)}_{\boldsymbol{m},\boldsymbol{m}+\boldsymbol{\alpha};\boldsymbol{\ell}+\alpha,\boldsymbol{\ell}} = \sum_{\boldsymbol{v}} U^{(\boldsymbol{\alpha})}_{\boldsymbol{m}\boldsymbol{v}}\left(\mathrm{i}\,v'_1 - \mathrm{e}^{-\mathrm{i}\,\epsilon}\sum_{i=2}^{n}v'_i - \mathrm{e}^{-\mathrm{i}\,\epsilon}\mu^2\right)\left(U^{(\boldsymbol{\alpha})}_{\boldsymbol{\ell}\boldsymbol{v}}\right)^{-1} \tag{J.13}$$

where

$$U^{(\boldsymbol{\alpha})}_{\boldsymbol{m}\boldsymbol{v}} = \prod_{i=1}^{n}U^{(\alpha_i)}_{m_i,\ell_i}\,. \tag{J.14}$$

and

$$\begin{aligned}
&\mathrm{i}\,v'_1 - \mathrm{e}^{-\mathrm{i}\,\epsilon}\sum_{i=2}^{n}v'_i - \mathrm{e}^{-\mathrm{i}\,\epsilon}\mu^2\\
=\;& \mathrm{i}\,\frac{4\Omega}{\theta}(2y_1 + 2\sigma\alpha_1 + 1) - \mathrm{e}^{-\mathrm{i}\,\epsilon}\sum_{i=2}^{n}\frac{4\Omega}{\theta}(2y_i + 2\sigma\alpha_i + 1) - \mathrm{e}^{-\mathrm{i}\,\epsilon}\mu^2\\
=\;& \frac{8\Omega}{\theta}\left(\mathrm{i}\,y_1 + \mathrm{i}\,(\sigma\alpha_1 + 1/2) - \mathrm{e}^{-\mathrm{i}\,\epsilon}\sum_{i=2}^{n}y_i - \mathrm{e}^{-\mathrm{i}\,\epsilon}\sum_{i=2}^{n}(\sigma\alpha_i + 1/2) - \mathrm{e}^{-\mathrm{i}\,\epsilon}\mu^2\frac{\theta}{8\Omega}\right)\,.
\end{aligned} \tag{J.15}$$

with $y_i = 0, 1, 2, \ldots$. One can show that

$$\left(U^{(\alpha_1)}_{m_1v_1}\cdots U^{(\alpha_n)}_{m_nv_n}\right)^{-1} = U^{(\alpha_n)}_{m_nv_n}\cdots U^{(\alpha_1)}_{m_1v_1}\,. \tag{J.16}$$

In the following we will use the notation $U^{(\alpha_i)}_{m_i v_i} = U^{(\alpha_i)}_{m_i}(y_i)$ where the relation between v_i and y_i is given by (J.15). Using the Schwinger parameter this yields the propagator

$$
\begin{aligned}
&\Delta^{(\epsilon)}_{\boldsymbol{m}+\boldsymbol{\alpha},\boldsymbol{m};\boldsymbol{\ell}+\boldsymbol{\alpha},\boldsymbol{\ell}} \\
= \; & -\left(\sum_{y_1=0}^{\infty}\cdots\sum_{y_n=0}^{\infty}\right)\int_0^\infty \mathrm{d}t\,\exp\left\{t(\mathrm{i}\,v_1' - \mathrm{e}^{-\mathrm{i}\,\epsilon}\sum_{2}^{n} v_i' - \mathrm{e}^{-\mathrm{i}\,\epsilon}\mu^2)\right\}\prod_{i=1}^{n}\left(U^{(\alpha_i)}_{m_i}(y_i)U^{(\alpha_i)}_{\ell_i}(y_i)\right) \\
= \; & -\mathrm{e}^{\,\mathrm{i}\,\epsilon}\frac{\theta}{8\Omega}\int_0^\infty \mathrm{d}t\;\mathrm{e}^{\,\mathrm{i}\,t\,\mathrm{e}^{\,\mathrm{i}\,\epsilon}(\sigma\alpha_1+1/2)-t\sum_{i=1}^{n-1}(\sigma\alpha_i+1/2)-t\frac{\theta\mu^2}{8\Omega}} \\
& \times\left(\sum_{y_1=0}^{\infty}\mathrm{e}^{\,\mathrm{i}\,t\,\mathrm{e}^{\,\mathrm{i}\,\epsilon}y_1}U^{(\alpha_1)}_{n_1}(y_1)U^{(\alpha_1)}_{\ell_1}(y_1)\right)\prod_{i=2}^{n}\left(\sum_{y_i=0}^{\infty}\mathrm{e}^{-ty_i}U^{(\alpha_i)}_{m_i}(y_i)U^{(\alpha_i)}_{\ell_i}(y_i)\right)
\end{aligned}
\tag{J.17}
$$

The only difference between the Euclidean and Minkowskian part is the additional factor "$-\mathrm{i}\,\mathrm{e}^{\,\mathrm{i}\,\epsilon}$" in the exponent of the y_1 part. We will consider the two factors depending on y_1 and y_i for $i = 2, \ldots, n$ separately. Using the explicit formula for the U's (J.10) the respective sums are given by

$$
\begin{aligned}
&\sum_{y^0=0}^{\infty}\mathrm{e}^{\,\mathrm{i}\,t\,\mathrm{e}^{\,\mathrm{i}\,\epsilon}y_1}U^{(\alpha_1)}_{m_1}(y_1)U^{(\alpha_0)}_{\ell_1}(y_1) \\
= \; & \sqrt{\begin{pmatrix}\alpha_1+m_1\\ m_1\end{pmatrix}\begin{pmatrix}\alpha_1+\ell_1\\ \ell_1\end{pmatrix}}\left(\frac{4\Omega}{(1+\Omega)^2}\right)^{\alpha_1+1}\left(\frac{1-\Omega}{1+\Omega}\right)^{m_1+\ell_1} \\
\times \; & \sum_{y_1=0}^{\infty}\begin{pmatrix}\alpha_1+y_1\\ y_1\end{pmatrix}\left(\frac{\mathrm{e}^{\,\mathrm{i}\,t\,\mathrm{e}^{\,\mathrm{i}\,\epsilon}}(1-\Omega)^2}{(1+\Omega)^2}\right)^{y_1} \\
\times \; & {}_2F_1\left(\begin{matrix}-m_1,-y_1\\ 1+\alpha_1\end{matrix}\,\middle|\,-\frac{4\Omega}{(1-\Omega)^2}\right){}_2F_1\left(\begin{matrix}-\ell_1,-y_1\\ 1+\alpha_1\end{matrix}\,\middle|\,-\frac{4\Omega}{(1-\Omega)^2}\right)
\end{aligned}
\tag{J.18}
$$

and

$$
\begin{aligned}
&\sum_{y_i=0}^{\infty}\mathrm{e}^{-ty_i}U^{(\alpha_i)}_{m_i}(y_i)U^{(\alpha_i)}_{\ell_i}(y_i) \\
= \; & \sqrt{\begin{pmatrix}\alpha_i+m_i\\ m_i\end{pmatrix}\begin{pmatrix}\alpha_i+\ell_i\\ \ell_i\end{pmatrix}}\left(\frac{4\Omega}{(1+\Omega)^2}\right)^{\alpha_i+1}\left(\frac{1-\Omega}{1+\Omega}\right)^{m_i+\ell_i} \\
\times \; & \sum_{y_i=0}^{\infty}\begin{pmatrix}\alpha_i+y_i\\ y_i\end{pmatrix}\left(\frac{\mathrm{e}^{-t}(1-\Omega)^2}{(1+\Omega)^2}\right)^{y_i} \\
\times \; & {}_2F_1\left(\begin{matrix}-m_i,-y_i\\ 1+\alpha_i\end{matrix}\,\middle|\,-\frac{4\Omega}{(1-\Omega)^2}\right){}_2F_1\left(\begin{matrix}-\ell_i,-y_i\\ 1+\alpha_i\end{matrix}\,\middle|\,-\frac{4\Omega}{(1-\Omega)^2}\right).
\end{aligned}
\tag{J.19}
$$

Now following Grosse & Wulkenhaar in [GW05b] we use the formula

$$
\begin{aligned}
&\sum_{y=0}^{\infty}a^y\begin{pmatrix}\alpha+y\\ y\end{pmatrix}{}_2F_1\left(\begin{matrix}-m,-y\\ 1+\alpha\end{matrix}\,\middle|\,b\right)2F_1\left(\begin{matrix}-l,-y\\ 1+\alpha\end{matrix}\,\middle|\,b\right) \\
= \; & \frac{(1-(1-b)a)^{m+l}}{(1-a)^{\alpha+m+l+1}}{}_2F_1\left(\begin{matrix}-m,-l\\ 1+\alpha\end{matrix}\,\middle|\,\frac{ab^2}{(1-(1-b)a)^2}\right), \quad |a|<1,
\end{aligned}
\tag{J.20}
$$

which can be applied both for the Euclidean as for the Minkowskian case, with

$$\begin{aligned}
a_1 &= \frac{\mathrm{e}^{\,\mathrm{i}\,t\,\mathrm{e}^{\,\mathrm{i}\,\epsilon}}(1-\Omega)^2}{(1+\Omega)^2}\\
b &= -\frac{4\Omega}{(1-\Omega)^2}\\
(1-b)a_1 &= \frac{(1+\Omega)^2}{(1-\Omega)^2}\frac{\mathrm{e}^{\,\mathrm{i}\,t\,\mathrm{e}^{\,\mathrm{i}\,\epsilon}}(1-\Omega)^2}{(1+\Omega)^2} = \mathrm{e}^{\,\mathrm{i}\,t\,\mathrm{e}^{\,\mathrm{i}\,\epsilon}}\\
(1-(1-b)a_1)^{m_1+\ell_1} &= (1-\mathrm{e}^{\,\mathrm{i}\,t\,\mathrm{e}^{\,\mathrm{i}\,\epsilon}})^{m_1+\ell_1}\\
\frac{a_1 b^2}{(1-(1-b)a_1)^2} &= \frac{(1-\Omega)^2}{(1+\Omega)^2}\left(\frac{4\Omega}{(1-\Omega)^2}\right)^2\frac{\mathrm{e}^{\,\mathrm{i}\,t\,\mathrm{e}^{\,\mathrm{i}\,\epsilon}}}{(1-\mathrm{e}^{\,\mathrm{i}\,t\,\mathrm{e}^{\,\mathrm{i}\,\epsilon}})^2}\\
&= \frac{(1+\Omega)^2}{(1-\Omega)^2}\left(\frac{4\Omega}{(1+\Omega)^2}\right)^2\frac{\mathrm{e}^{\,\mathrm{i}\,t\,\mathrm{e}^{\,\mathrm{i}\,\epsilon}}}{(1-\mathrm{e}^{\,\mathrm{i}\,t\,\mathrm{e}^{\,\mathrm{i}\,\epsilon}})^2}
\end{aligned} \tag{J.21}$$

and

$$\begin{aligned}
a_i &= \frac{\mathrm{e}^{-t}(1-\Omega)^2}{(1+\Omega)^2}\\
b &= -\frac{4\Omega}{(1-\Omega)^2}\\
(1-b)a_i &= \frac{(1+\Omega)^2}{(1-\Omega)^2}\frac{\mathrm{e}^{-t}(1-\Omega)^2}{(1+\Omega)^2} = \mathrm{e}^{-t}\\
(1-(1-b)a_i)^{m_i+\ell_i} &= (1-\mathrm{e}^{-t})^{m_i+\ell_i}\\
\frac{a_i b^2}{(1-(1-b)a_i)^2} &= \frac{(1-\Omega)^2}{(1+\Omega)^2}\left(\frac{4\Omega}{(1-\Omega)^2}\right)^2\frac{\mathrm{e}^{-t}}{(1-\mathrm{e}^{-t})^2}\\
&= \frac{(1+\Omega)^2}{(1-\Omega)^2}\left(\frac{4\Omega}{(1+\Omega)^2}\right)^2\frac{\mathrm{e}^{-t}}{(1-\mathrm{e}^{-t})^2}
\end{aligned} \tag{J.22}$$

Inserting the above expressions leads to

$$\begin{aligned}
&\frac{(1-\mathrm{e}^{\,\mathrm{i}\,t\,\mathrm{e}^{\,\mathrm{i}\,\epsilon}})^{m_1+\ell_1}}{\left(1-\frac{\mathrm{e}^{\,\mathrm{i}\,t\,\mathrm{e}^{\,\mathrm{i}\,\epsilon}}(1-\Omega)^2}{(1+\Omega)^2}\right)^{\alpha_1+m_1+\ell_1+1}}\sqrt{\begin{pmatrix}\alpha_1+m_1\\ m_1\end{pmatrix}\begin{pmatrix}\alpha_1+\ell_1\\ \ell_1\end{pmatrix}}\\
\times\quad &{}_2F_1\left(\begin{matrix}-m_1,-\ell_1\\ 1+\alpha_1\end{matrix}\,\middle|\,\left(\frac{4\Omega}{(1+\Omega)^2}\right)^2\left(\frac{1+\Omega}{1-\Omega}\right)^2\frac{\mathrm{e}^{\,\mathrm{i}\,t}}{(1-\mathrm{e}^{\,\mathrm{i}\,t})^2}\right)
\end{aligned} \tag{J.23}$$

and

$$\begin{aligned}
&\frac{(1-\mathrm{e}^{-t})^{m_i+\ell_i}}{\left(1-\frac{\mathrm{e}^{-t}(1-\Omega)^2}{(1+\Omega)^2}\right)^{\alpha_i+m_i+\ell_i+1}}\sqrt{\begin{pmatrix}\alpha_i+m_i\\ m_i\end{pmatrix}\begin{pmatrix}\alpha_i+\ell_i\\ \ell_i\end{pmatrix}}\\
\times\quad &{}_2F_1\left(\begin{matrix}-m_i,-\ell_i\\ 1+\alpha_i\end{matrix}\,\middle|\,\left(\frac{4\Omega}{(1+\Omega)^2}\right)^2\left(\frac{1+\Omega}{1-\Omega}\right)^2\frac{\mathrm{e}^{-t}}{(1-\mathrm{e}^{-t})^2}\right).
\end{aligned} \tag{J.24}$$

Now substituting $z = \mathrm{e}^{-t}$ (which gives a z^{-1} from the differential) and using the expansion of the hypergeometric functions

$${}_2F_1\left(\begin{matrix}-m,-l\\ 1+\alpha\end{matrix}\,\middle|\,A\right) = \sum_{u=\max(0,-\alpha)}^{\min(m,\ell)} A^u\frac{m!\ell!\alpha!}{(m-u)!(\ell-u)!(\alpha+u)!u!} \tag{J.25}$$

and recombining the binomials and faculties

$$\begin{aligned}
&\sqrt{\left(\begin{array}{c}\alpha+n\\ n\end{array}\right)\left(\begin{array}{c}\alpha+\ell\\ \ell\end{array}\right)}\frac{n!\ell!\alpha!}{(n-u)!(\ell-u)!(\alpha+u)!u!}\\
=&\sqrt{\frac{(\alpha+n)!(\alpha+\ell)!n!n!\ell!\ell!\alpha!\alpha!}{n!\alpha!\ell!\alpha!(n-u)!(n-u)!(\ell-u)!(\ell-u)!(\alpha+u)!(\alpha+u)!u!u!}}\\
=&\sqrt{\left(\frac{(\alpha+n)!}{(\alpha+u)!(n-u)!}\right)\left(\frac{(\alpha+\ell)!}{(\alpha+u)!(\ell-u)!}\right)\left(\frac{n!}{(n-u)!u!}\right)\left(\frac{\ell!}{(l-u)!u!}\right)}\\
=&\sqrt{\left(\begin{array}{c}\alpha+n\\ \alpha+u\end{array}\right)\left(\begin{array}{c}\alpha+\ell\\ \alpha+u\end{array}\right)\left(\begin{array}{c}n\\ u\end{array}\right)\left(\begin{array}{c}\ell\\ u\end{array}\right)}\\
=&\mathcal{A}(m,\ell,\alpha,u)
\end{aligned}\tag{J.26}$$

this becomes

$$\sum_{u_1=\max(0,-\alpha_1)}^{\min(m_1,\ell_1)}\frac{z^{-\mathrm{i}\,\mathrm{e}^{\,\mathrm{i}\,\epsilon}u_1}(1-z^{-i\,\mathrm{e}^{\,\mathrm{i}\,\epsilon}})^{m_1+\ell_1-2u_1}}{\left(1-\frac{(1-\Omega)^2}{(1+\Omega)^2}z^{-\mathrm{i}}\right)^{\alpha_1+m_1+\ell_1+1}}\left(\frac{4\Omega}{(1+\Omega)^2}\right)^{\alpha_1+2u_1+1}\left(\frac{1-\Omega}{1+\Omega}\right)^{m_1+\ell_1-2u_1}\mathcal{A}(m_1,\ell_1,\alpha_1,u_1)\tag{J.27}$$

and

$$\sum_{u_i=\max(0,-\alpha_i)}^{\min(m_i,\ell_i)}\frac{z^{u_i}(1-z)^{m_i+\ell_i-2u_i}}{\left(1-\frac{(1-\Omega)^2}{(1+\Omega)^2}z\right)^{\alpha_i+m_i+\ell_i+1}}\left(\frac{4\Omega}{(1+\Omega)^2}\right)^{\alpha_i+2u_i+1}\left(\frac{1-\Omega}{1+\Omega}\right)^{m_i+\ell_i-2u_i}\mathcal{A}(m_i,\ell_i,\alpha_i,u_i)\,.\tag{J.28}$$

This proves the theorem. □

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